J. Inequal. Math. Anal., 1(2), 2025, 79–96

Journal of Inequalities and Mathematical Analysis

ISSN:3062-3251

Journal Homepage: www.inequalmath.com

Research Article

Some New Local Fractional Newton-type Inequalities

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Abstract

In this study, we present a novel identity based on local fractional integrals, which forms the basis for deriving a series of new Newton-type inequalities for functions whose local fractional derivatives demonstrate generalized convexity. Additionally, we establish further results by employing the generalized Hölder inequality, the generalized power mean inequality, and an improved version of the generalized power mean inequality, along with another result for local fractional differentiable concave functions. To substantiate our theoretical findings, we include several practical applications that highlight the effectiveness and applicability of the derived results.

Keywords: Fractal sets, Newton-type inequalities, generalized convex functions, generalized Hölder inequality, generalized power mean inequality

2020 MSC: 26D10, 26D15, 26A51.

1. Introduction

Convexity is a fundamental mathematical concept that plays a pivotal role in various fields, including optimization, analysis, and geometry.

Definition 1.1. Consider $\hbar : I \subseteq \mathbb{R} \to \mathbb{R}$. If, for any $s_1, s_2 \in I$ and $t \in [0,1]$, the inequality

$$\hbar(ts_1 + (1-t)s_2) \leq t\hbar(s_1) + (1-t)\hbar(s_2)$$

is satisfied, then $\hbar$ is a convex function on I, see [1].

The theory of inequalities is recognized for its significant impact across a wide spectrum of both pure and applied sciences. The application of convexity in inequalities allows for the formulation of powerful results and techniques, contributing to a deeper understanding of mathematical structures and their applications. This symbiotic relationship between convexity and the theory of inequalities highlights the pervasive influence of convexity in almost all areas

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doi: 10.63286/jima.008

Received: 20 February 2025 Revised: 15 April 2025 Accepted: 17 May 2025

of pure and applied sciences. The most significant result arising from this combination is undoubtedly the Hermite-Hadamard inequality, presented as follows:

$$\hbar\left(\tfrac{s_1+s_2}{2}\right) \leq \frac{1}{s_2-s_1}\int_{s_1}^{s_2}\hbar(s)ds \leq \tfrac{\hbar(s_1)+\hbar(s_2)}{2},$$

where $\hbar$ is convex on $[s_1,s_2]$.

Note that this mathematical tool is extensively employed for estimating the error of quadrature formulas [2, 3]. Notably, among the well-known and widely used quadratures are those of Simpson. The following inequality is recognized as the Newton-type inequality for four-times continuously differentiable functions:

$$\left|\tfrac{1}{8}\left(\hbar(s_1)+3\hbar\left(\tfrac{2s_1+s_2}{3}\right)+3\hbar\left(\tfrac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\tfrac{1}{s_2-s_1}\int_{s_1}^{s_2}\hbar(s)\,ds\right| \leq \tfrac{(s_2-s_1)^4}{6480}\left\|\hbar^{(4)}\right\|_\infty .$$

In [4], the authors presented the following Newton-type inequalities involving first-order derivatives:

$$\begin{aligned}&\left|\tfrac{1}{8}\left(\hbar(s_1)+3\hbar\left(\tfrac{2s_1+s_2}{3}\right)+3\hbar\left(\tfrac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\tfrac{1}{s_2-s_1}\int_{s_1}^{s_2}\hbar(s)\,ds\right|\\ &\leq \tfrac{25(s_2-s_1)}{576}\left(\left|\hbar'(s_1)\right|+\left|\hbar'(s_2)\right|\right),\end{aligned}$$

where $|\hbar'|$ is convex on $[s_1,s_2]$.

In [5], Laribi et al. provided a refinement of the above-mentioned result for the same class of functions as follows:

$$\begin{aligned}&\left|\tfrac{1}{8}\left(\hbar(s_1)+3\hbar\left(\tfrac{2s_1+s_2}{3}\right)+3\hbar\left(\tfrac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\tfrac{1}{s_2-s_1}\int_{s_1}^{s_2}\hbar(s)\,ds\right|\\ &\leq \tfrac{s_2-s_1}{9}\left(\tfrac{157}{1536}\left(\left|\hbar'(s_1)\right|+\left|\hbar'(s_2)\right|\right)+\tfrac{443}{1536}\left(\left|\hbar'\left(\tfrac{2s_1+s_2}{3}\right)\right|+\left|\hbar'\left(\tfrac{s_1+2s_2}{3}\right)\right|\right)\right).\end{aligned}$$

For further studies on Newton-type inequalities via different type of integrals, we refer the readers to [6–10].

The groundbreaking framework of local fractional calculus, initially introduced by Yang [11, 12], provides a powerful tool for addressing non-differentiable challenges prevalent in intricate real-world systems. It finds particular utility in the modeling of non-differentiable phenomena in science and engineering through the formulation of ordinary or partial differential equations featuring local fractional expressions. Consequently, a host of researchers, specializing in areas like mathematical physics and applied sciences [13, 14], have dedicated their efforts to exploring these nuanced topics. Yang's extensive contributions encompass a diverse array of subjects within the realm of local fractional calculus and its practical applications. Furthermore, the mathematical community has also delved into the study of various inequalities, recognizing their pivotal role in advancing mathematical understanding.

In terms of local fractional integral inequalities, notable contributions have been made by researchers. Chen, for instance, established the generalized Hölder's inequality on a fractal space in [15]. Mo et al., in [16], proved the generalized Hermite-Hadamard inequality for generalized convex functions. Sarikaya et al., as documented in [17], presented some Simpson's type integral inequalities for mappings with generalized convex local fractional derivatives. Further developments include the introduction of Bullen-type inequalities by Yu et al. in [18], Simpson-type in [19], Milne-type in [20], and Companion Ostrowski in [21]. For an in-depth exploration of works in this direction, interested readers are encouraged to consult [22–27].

In [28], the authors established the following inequality of Newton-type for local fractional integrals.

$$\begin{aligned}&\left|\tfrac{1}{8^\rho}\left(\hbar(s_1)+3^\rho\hbar\left(\tfrac{2s_1+s_2}{3}\right)+3^\rho\hbar\left(\tfrac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\tfrac{\Gamma(\rho+1)}{(s_2-s_1)^\rho}\,{}_{s_1}I_{s_2}^{\rho}\hbar(s)\right|\\ &\leq (s_2-s_1)^\rho\left(\left(\tfrac{528}{13824}\right)^\rho\tfrac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}+\left(\tfrac{1008}{13824}\right)^\rho\tfrac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}\right)\left(\left|\hbar^{(\rho)}(s_1)\right|+\left|\hbar^{(\rho)}(s_2)\right|\right),\end{aligned}$$

where $\left|\hbar^{(\rho)}\right|$ is a generalized convex function.

However, the above result represents a broad estimate as it only involves the endpoints of the interval, and it would be more interesting to establish an estimate involving evaluations of $\hbar^{(\rho)}$ at multiple points within the interval $[s_1,s_2]$.

Motivated by the above-mentioned works, especially the research presented in [28], in this study, we establish some new Newton-type inequalities via generalized convexity. The exploration is rounded off with several applications, illustrating the practical implications of our findings.

2. Preliminaries

In this section, we provide the foundational definitions, propositions, and lemmas necessary to understand the concept of local calculus as introduced by Yang [11, 12]. We begin by defining key terms and concepts essential to our subsequent discussions.

Definition 2.1 ([11])**.** A non-differentiable function $\hbar : \mathbb{R} \to \mathbb{R}^\rho$ is local fractional continuous at s_0 , if

$$\forall \varepsilon > 0, \exists \delta > 0 : |\hbar(s) - \hbar(s_0)| < \varepsilon^\rho$$

holds for $|s - s_0| < \delta$. We denote the set of all local fractional continuous functions on (s_1,s_2) by $C_\rho\,(s_1,s_2)$.

Definition 2.2 ([11])**.** The local fractional derivative of $\hbar(s)$ of order ρ at $s = s_0$ is defined as:

$$\hbar^{(\rho)}(s_0) = \left.\frac{d^\rho \hbar(s)}{ds^\rho}\right|_{s=s_0} = \lim_{s\to s_0} \frac{\Delta^\rho(\hbar(s)-\hbar(s_0))}{(s-s_0)^\rho},$$

where $\Delta^\rho\,(\hbar(s) - \hbar(s_0)) \cong \Gamma(\rho+1)\,(\hbar(s) - \hbar(s_0))$.

If $\hbar^{k\rho}(s) = \underbrace{D^\rho D^\rho ... D^\rho}_{k \text{ times}} \hbar(s)$ exists for any $s \in I \subseteq R$, then we denote $\hbar \in D_{k\rho}(I)$, with $k = 0,1,2,3,....$

Definition 2.3 ([11])**.** Consider $\hbar(x) \in C_\rho\,[s_1,s_2]$. Then the local fractional integral is defined by,

$$_{s_1}I^{\rho}_{s_2}\hbar(s) = \frac{1}{\Gamma(\rho+1)}\int_{s_1}^{s_2} \hbar(\sigma)\,(d\sigma)^\rho = \frac{1}{\Gamma(\rho+1)} \lim_{\Delta\sigma\to 0} \sum_{n=0}^{N-1} \hbar(\sigma_j)\,(\Delta\sigma_j)^\rho,$$

where $\Delta\sigma_n = \sigma_{n+1} - \sigma_n$ and $\Delta\sigma = \max\{\Delta\sigma_1, \Delta\sigma_2, ..., \Delta\sigma_{N-1}\}$, where $[\sigma_n, \sigma_{n+1}], n = 0,1,...,N-1$ and $s_1 = \sigma_0 < \sigma_1 < ... < \sigma_N = s_2$ is partition of the interval $[s_1,s_2]$.

In this context, it is evident that $_{s_1}I^{\rho}_{s_2}\hbar(s) = 0$ when $s_1 = s_2$, and $_{s}1I^{\rho}_{s_2}\hbar(s) = -_{s_2}I^{\rho}_{s_1}\hbar(s)$ when $s_1 < s_2$. If $_{s_1}I^{\rho}_{s_2}\hbar(s)$ exists for any $s \in [s_1,s_2]$, then we characterize $\hbar(s) \in I^{\rho}_{s}\,[s_1,s_2]$.

Lemma 2.4 ([11])**.** *1. Suppose that* $\hbar(s) = \wp^{(\rho)}(s) \in C_\rho\,[s_1,s_2]$ *, then we have*

$$_{s_1}I^{\rho}_{s_2}\hbar(s) = \wp(s_2) - \wp(s_1).$$

2. Suppose that $\hbar, \wp \in D_\rho\,[s_1,s_2]$ *and* $\hbar^{(\rho)}(s), \wp^{(\rho)}(s) \in C_\rho\,[s_1,s_2]$*, then we have*

$$_{s_1}I^{\rho}_{s_2}\hbar(s)\,\wp^{(\rho)}(s) = \hbar(s)\,\wp(s)|_{s_1}^{s_2} - {}_{s_1}I^{\rho}_{s_2}\hbar^{(\rho)}(s)\,\wp(s).$$

Lemma 2.5 ([11])**.** *For* $z \in \mathbb{R}$ *, we have*

$$\frac{d^\rho s^{z\rho}}{ds^\rho} = \frac{\Gamma(1+z\rho)}{\Gamma(1+(z-1)\rho)} s^{(z-1)\rho},$$

$$\frac{1}{\Gamma(1+\rho)}\int_{s_1}^{s_2} s^{z\rho}\,(ds)^\rho = \frac{\Gamma(1+z\rho)}{\Gamma(1+(z+1)\rho)}\left(s_2^{(z+1)\rho} - s_1^{(z+1)\rho}\right).$$

Lemma 2.6 (Generalized Hölder's inequality [15]). *Let $\hbar,\wp\in C_\rho[s_1,s_2]$ and $|\hbar(s)|^p, |\wp(s)|^q$ where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, be both integrable under the frame of the fractal spaces, then we have*

$$\frac{1}{\Gamma(1+\rho)}\int_{s_1}^{s_2}|\hbar(s)\wp(s)|(ds)^\rho \le \left(\frac{1}{\Gamma(1+\rho)}\int_{s_1}^{s_2}|\hbar(s)|^p(ds)^\rho\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\rho)}\int_{s_1}^{s_2}|\wp(s)|^q(ds)^\rho\right)^{\frac{1}{q}}.$$

Lemma 2.7 (Improved generalized power mean inequality [29]). *Let $\hbar,\wp\in C_\rho[s_1,s_2]$ and $|\hbar(s)|, |\hbar(s)||\wp(s)|^q$ where $q>1$ are both integrable within the context of fractal spaces, then we have*

$$\begin{aligned}&\frac{1}{\Gamma(1+\rho)}\int_{s_1}^{s_2}|\hbar(s)\wp(s)|(ds)^\rho\\ &\le\left(\frac{1}{s_2-s_1}\right)^\rho\left(\left(\frac{1}{\Gamma(1+\rho)}\int_{s_1}^{s_2}(s_2-s)^\rho|\hbar(s)|(ds)^\rho\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(1+\rho)}\int_{s_1}^{s_2}(s_2-s)^\rho|\hbar(s)||\wp(s)|^q(ds)^\rho\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\frac{1}{\Gamma(1+\rho)}\int_{s_1}^{s_2}(s-s_1)^\rho|\hbar(s)|(ds)^\rho\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(1+\rho)}\int_{s_1}^{s_2}(s-s_1)^\rho|\hbar(s)||\wp(s)|^q(ds)^\rho\right)^{\frac{1}{q}}\right).\end{aligned}$$

Definition 2.8 (Generalized convex function [11]). Let $\hbar: I\subseteq\mathbb{R}\to\mathbb{R}^\rho$. For any $s_1,s_2\in I$ and $t\in[0,1]$, if

$$\hbar(ts_1+(1-t)s_2)\le t^\rho\hbar(s_1)+(1-t)^\rho\hbar(s_2)$$

holds, then $\hbar$ is considered a generalized convex function on I. Conversely, if the inequality holds in the opposite direction, then the function $\hbar$ is referred to as generalized concave.

3. Main results

The subsequent lemma will play a pivotal and fundamental role in substantiating the outcomes of our study.

Lemma 3.1. *Let $\hbar: I=[s_1,s_2]\to\mathbb{R}^\rho$ be a function such that $\hbar\in D_\rho(I^\circ)$, and $\hbar^{(\rho)}\in C_\rho[s_1,s_2]$, then the following equality holds for all $\rho>0$.*

$$\begin{aligned}&\frac{1}{8^\rho}\left(\hbar(s_1)+3^\rho\hbar\left(\frac{2s_1+s_2}{3}\right)+3^\rho\hbar\left(\frac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\frac{\Gamma(\rho+1)}{(s_2-s_1)^\rho}\,{}_{s_1}I_{s_2}^{\rho}\hbar(s)\\ &=\frac{(s_2-s_1)^\rho}{9^\rho}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left(t-\frac{3}{8}\right)^\rho\hbar^{(\rho)}\left((1-t)s_1+t\frac{2s_1+s_2}{3}\right)(dt)^\rho\right.\\ &-\frac{1}{\Gamma(\rho+1)}\int_0^1\left(\frac{1}{2}-t\right)^\rho\hbar^{(\rho)}\left((1-t)\frac{2s_1+s_2}{3}+t\frac{s_1+2s_2}{3}\right)(dt)^\rho\\ &\left.+\frac{1}{\Gamma(\rho+1)}\int_0^1\left(t-\frac{5}{8}\right)^\rho\hbar^{(\rho)}\left((1-t)\frac{s_1+2s_2}{3}+ts_2\right)(dt)^\rho\right),\end{aligned}$$

where

$${}_{s_1}I_{s_2}^{\rho}\hbar(s)=\frac{1}{\Gamma(\rho+1)}\int_{s_1}^{\frac{2s_1+s_2}{3}}\hbar(s)(ds)^\rho+\frac{1}{\Gamma(\rho+1)}\int_{\frac{2s_1+s_2}{3}}^{\frac{s_1+2s_2}{3}}\hbar(s)(ds)^\rho+\frac{1}{\Gamma(\rho+1)}\int_{\frac{s_1+2s_2}{3}}^{s_2}\hbar(s)(ds)^\rho. \quad (3.1)$$

Proof. Let

$$\mathscr{I} = \mathscr{I}_1 - \mathscr{I}_2 + \mathscr{I}_3, \tag{3.2}$$

with

$$\mathscr{I}_1 = \frac{1}{\Gamma(\rho+1)} \int_0^1 \left(t - \frac{3}{8}\right)^\rho \hbar^{(\rho)}\left((1-t)s_1 + t\frac{2s_1+s_2}{3}\right)(dt)^\rho,$$

$$\mathscr{I}_2 = \frac{1}{\Gamma(\rho+1)} \int_0^1 \left(\frac{1}{2} - t\right)^\rho \hbar^{(\rho)}\left((1-t)\frac{2s_1+s_2}{3} + t\frac{s_1+2s_2}{3}\right)(dt)^\rho$$

and

$$\mathscr{I}_3 = \frac{1}{\Gamma(\rho+1)} \int_0^1 \left(t - \frac{5}{8}\right)^\rho \hbar^{(\rho)}\left((1-t)\frac{s_1+2s_2}{3} + ts_2\right)(dt)^\rho.$$

Using Lemma 2.4, $\mathscr{I}_1$ gives

$$\begin{aligned}\mathscr{I}_1 &= \frac{3^\rho}{(s_2-s_1)^\rho}\left(t - \frac{3}{8}\right)^\rho \hbar\left((1-t)s_1 + t\frac{2s_1+s_2}{3}\right)\Big|_0^1 - \frac{3^\rho\Gamma(\rho+1)}{(s_2-s_1)^\rho}\int_0^1 \hbar\left((1-t)s_1 + t\frac{2s_1+s_2}{3}\right)(dt)^\rho \\ &= \frac{15^\rho}{8^\rho(s_2-s_1)^\rho}\hbar\left(\frac{2s_1+s_2}{3}\right) + \frac{9^\rho}{8^\rho(s_2-s_1)^\rho}\hbar(s_1) - \frac{9^\rho\Gamma(\rho+1)}{(s_2-s_1)^{2\rho}\Gamma(\rho+1)}\int_{s_1}^{\frac{2s_1+s_2}{3}} \hbar(s)(ds)^\rho.\end{aligned} \tag{3.3}$$

Similarly, we obtain

$$\begin{aligned}\mathscr{I}_2 &= \frac{3^\rho}{(s_2-s_1)^\rho}\left(\frac{1}{2} - t\right)^\rho \hbar\left((1-t)\frac{2s_1+s_2}{3} + t\frac{s_1+2s_2}{3}\right)\Big|_0^1 + \frac{3^\rho\Gamma(\rho+1)}{(s_2-s_1)^\rho}\int_0^1 \hbar\left((1-t)\frac{2s_1+s_2}{3} + t\frac{s_1+2s_2}{3}\right)(dt)^\rho \\ &= -\frac{3^\rho}{2^\rho(s_2-s_1)^\rho}\hbar\left(\frac{s_1+2s_2}{3}\right) - \frac{3^\rho}{2^\rho(s_2-s_1)^\rho}\hbar\left(\frac{2s_1+s_2}{3}\right) + \frac{9^\rho\Gamma(\rho+1)}{(s_2-s_1)^{2\rho}\Gamma(\rho+1)}\int_{\frac{2s_1+s_2}{3}}^{\frac{s_1+2s_2}{3}} \hbar(s)(ds)^\rho\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\mathscr{I}_3 &= \frac{3^\rho}{(s_2-s_1)^\rho}\left(t - \frac{5}{8}\right)^\rho \hbar\left((1-t)\frac{s_1+2s_2}{3} + ts_2\right)\Big|_0^1 - \frac{3^\rho\Gamma(\rho+1)}{(s_2-s_1)^\rho}\int_0^1 \hbar\left((1-t)\frac{s_1+2s_2}{3} + ts_2\right)(dt)^\rho \\ &= \frac{9^\rho}{8^\rho(s_2-s_1)^\rho}\hbar(s_2) + \frac{15^\rho}{8^\rho(s_2-s_1)^\rho}\hbar\left(\frac{s_1+2s_2}{3}\right) - \frac{9^\rho\Gamma(\rho+1)}{(s_2-s_1)^{2\rho}\Gamma(\rho+1)}\int_{\frac{s_1+2s_2}{3}}^{s_2} \hbar(s)(ds)^\rho.\end{aligned} \tag{3.5}$$

By substituting equalities (3.3)-(3.5) into (3.2), and subsequently multiplying the obtained equality by $\frac{(s_2-s_1)^\rho}{9^\rho}$, we achieve the desired outcome. □

Theorem 3.2. *Let $\hbar : I = [s_1, s_2] \to \mathbb{R}^\rho$ be a function such that $\hbar \in D_\rho(I^\circ)$, and $\hbar^{(\rho)} \in C_\rho[s_1, s_2]$ with $s_1 < s_2$. If $\left|\hbar^{(\rho)}\right|$ is generalized convex on $[s_1, s_2]$, then we have*

$$\left|\frac{1}{8^\rho}\left(\hbar(s_1) + 3^\rho\hbar\left(\frac{2s_1+s_2}{3}\right) + 3^\rho\hbar\left(\frac{s_1+2s_2}{3}\right) + \hbar(s_2)\right) - \frac{\Gamma(\rho+1)}{(s_2-s_1)^\rho}\, {}_{s_1}I_{s_2}^\rho \hbar(s)\right|$$

$$\leq \frac{(s_2-s_1)^\rho}{9^\rho}\Big(\Big(\big(\tfrac{131}{256}\big)^\rho \tfrac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)} - \big(\tfrac{35}{256}\big)^\rho \tfrac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\Big)\Big(\left|\hbar^{(\rho)}(s_1)\right| + \left|\hbar^{(\rho)}(s_2)\right|\Big) + \Big(\big(\tfrac{421}{256}\big)^\rho \tfrac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)} - \big(\tfrac{133}{256}\big)^\rho \tfrac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\Big)\Big(\left|\hbar^{(\rho)}\big(\tfrac{2s_1+s_2}{3}\big)\right| + \left|\hbar^{(\rho)}\big(\tfrac{s_1+2s_2}{3}\big)\right|\Big)\Big).$$

Proof. From Lemma 3.1 and properties of modulus, we have

$$\begin{aligned}
&\left|\tfrac{1}{8^\rho}\big(\hbar(s_1) + 3^\rho \hbar\big(\tfrac{2s_1+s_2}{3}\big) + 3^\rho \hbar\big(\tfrac{s_1+2s_2}{3}\big) + \hbar(s_2)\big) - \tfrac{\Gamma(\rho+1)}{(s_2-s_1)^\rho}\, {}_{s_1}I^{\rho}_{s_2}\hbar(s)\right| \\
&\leq \frac{(s_2-s_1)^\rho}{9^\rho}\Bigg(\frac{1}{\Gamma(\rho+1)}\int_0^1 \left|t-\tfrac{3}{8}\right|^\rho \left|\hbar^{(\rho)}\big((1-t)s_1 + t\tfrac{2s_1+s_2}{3}\big)\right|(dt)^\rho \\
&\quad + \frac{1}{\Gamma(\rho+1)}\int_0^1 \left|\tfrac{1}{2}-t\right|^\rho \left|\hbar^{(\rho)}\big((1-t)\tfrac{2s_1+s_2}{3} + t\tfrac{s_1+2s_2}{3}\big)\right|(dt)^\rho \\
&\quad + \frac{1}{\Gamma(\rho+1)}\int_0^1 \left|t-\tfrac{5}{8}\right|^\rho \left|\hbar^{(\rho)}\big((1-t)\tfrac{s_1+2s_2}{3} + ts_2\big)\right|(dt)^\rho\Bigg).
\end{aligned} \tag{3.6}$$

Since $\left|\hbar^{(\rho)}\right|$ is generalized convex, (3.6) yields

$$\begin{aligned}
&\left|\tfrac{1}{8^\rho}\big(\hbar(s_1) + 3^\rho \hbar\big(\tfrac{2s_1+s_2}{3}\big) + 3^\rho \hbar\big(\tfrac{s_1+2s_2}{3}\big) + \hbar(s_2)\big) - \tfrac{\Gamma(\rho+1)}{(s_2-s_1)^\rho}\, {}_{s_1}I^{\rho}_{s_2}\hbar(s)\right| \\
&\leq \frac{(s_2-s_1)^\rho}{9^\rho}\Bigg(\frac{1}{\Gamma(\rho+1)}\int_0^1 \left|t-\tfrac{3}{8}\right|^\rho \Big((1-t)^\rho\left|\hbar^{(\rho)}(s_1)\right| + t^\rho\left|\hbar^{(\rho)}\big(\tfrac{2s_1+s_2}{3}\big)\right|\Big)(dt)^\rho \\
&\quad + \frac{1}{\Gamma(\rho+1)}\int_0^1 \left|\tfrac{1}{2}-t\right|^\rho \Big((1-t)^\rho\left|\hbar^{(\rho)}\big(\tfrac{2s_1+s_2}{3}\big)\right| + t^\rho\left|\hbar^{(\rho)}\big(\tfrac{s_1+2s_2}{3}\big)\right|\Big)(dt)^\rho \\
&\quad + \frac{1}{\Gamma(\rho+1)}\int_0^1 \left|t-\tfrac{5}{8}\right|^\rho \Big((1-t)^\rho\left|\hbar^{(\rho)}\big(\tfrac{s_1+2s_2}{3}\big)\right| + t^\rho\left|\hbar^{(\rho)}(s_2)\right|\Big)(dt)^\rho\Bigg) \\
&= \frac{(s_2-s_1)^\rho}{9^\rho}\Big(\Big(\big(\tfrac{131}{256}\big)^\rho \tfrac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)} - \big(\tfrac{35}{256}\big)^\rho \tfrac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\Big)\Big(\left|\hbar^{(\rho)}(s_1)\right| + \left|\hbar^{(\rho)}(s_2)\right|\Big) \\
&\quad + \Big(\big(\tfrac{421}{256}\big)^\rho \tfrac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)} - \big(\tfrac{133}{256}\big)^\rho \tfrac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\Big)\Big(\left|\hbar^{(\rho)}\big(\tfrac{2s_1+s_2}{3}\big)\right| + \left|\hbar^{(\rho)}\big(\tfrac{s_1+2s_2}{3}\big)\right|\Big)\Big),
\end{aligned}$$

where we have used

$$\begin{aligned}
\frac{1}{\Gamma(\rho+1)}\int_0^1 \left|t-\tfrac{5}{8}\right|^\rho t^\rho (dt)^\rho &= \frac{1}{\Gamma(\rho+1)}\int_0^1 \left|t-\tfrac{3}{8}\right|^\rho (1-t)^\rho (dt)^\rho \\
&= \big(\tfrac{131}{256}\big)^\rho \tfrac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)} - \big(\tfrac{35}{256}\big)^\rho \tfrac{\Gamma(1+\rho)}{\Gamma(1+2\rho)},
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
\frac{1}{\Gamma(\rho+1)}\int_0^1 \left|t-\tfrac{3}{8}\right|^\rho t^\rho (dt)^\rho &= \frac{1}{\Gamma(\rho+1)}\int_0^1 \left|t-\tfrac{5}{8}\right|^\rho (1-t)^\rho (dt)^\rho \\
&= \big(\tfrac{229}{256}\big)^\rho \tfrac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)} - \big(\tfrac{69}{256}\big)^\rho \tfrac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}
\end{aligned} \tag{3.8}$$

and

$$\frac{1}{\Gamma(\rho+1)}\int_0^1 \left|\tfrac{1}{2}-t\right|^{\rho} t^{\rho}\,(dt)^{\rho} = \frac{1}{\Gamma(\rho+1)}\int_0^1 \left|\tfrac{1}{2}-t\right|^{\rho} (1-t)^{\rho}\,(dt)^{\rho}$$
$$= \left(\tfrac{3}{4}\right)^{\rho}\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)} - \left(\tfrac{1}{4}\right)^{\rho}\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}. \tag{3.9}$$

This concludes the demonstration. □

Corollary 3.3. *In Theorem 3.2, if $\rho \to 1$, we obtain*

$$\left|\tfrac{1}{8}\left(\hbar(s_1)+3\hbar\left(\tfrac{2s_1+s_2}{3}\right)+3\hbar\left(\tfrac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\tfrac{1}{s_2-s_1}\int_{s_1}^{s_2}\hbar(s)\,ds\right|$$
$$\le \tfrac{s_2-s_1}{9}\left(\tfrac{157}{1536}\left(\left|\hbar'(s_1)\right|+\left|\hbar'(s_2)\right|\right)+\tfrac{443}{1536}\left(\left|\hbar'\left(\tfrac{2s_1+s_2}{3}\right)\right|+\left|\hbar'\left(\tfrac{s_1+2s_2}{3}\right)\right|\right)\right),$$

which is the same result given in Corollary 2.1 from [5].

Corollary 3.4. *In Theorem 3.2, using the generalized convexity of $\left|\hbar^{(\rho)}\right|$, we get*

$$\left|\tfrac{1}{8^{\rho}}\left(\hbar(s_1)+3^{\rho}\hbar\left(\tfrac{2s_1+s_2}{3}\right)+3^{\rho}\hbar\left(\tfrac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\tfrac{\Gamma(\rho+1)}{(s_2-s_1)^{\rho}}\,{}_{s_1}I^{\rho}_{s_2}\hbar(s)\right|$$
$$\le \tfrac{(s_2-s_1)^{\rho}}{9^{\rho}}\left(\left(\tfrac{552}{256}\right)^{\rho}\tfrac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}-\left(\tfrac{168}{256}\right)^{\rho}\tfrac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right)\left(\left|\hbar^{(\rho)}(s_1)\right|+\left|\hbar^{(\rho)}(s_2)\right|\right).$$

Corollary 3.5. *In Corollary 3.4, if $\rho \to 1$, we obtain*

$$\left|\tfrac{1}{8}\left(\hbar(s_1)+3\hbar\left(\tfrac{2s_1+s_2}{3}\right)+3\hbar\left(\tfrac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\tfrac{1}{s_2-s_1}\int_{s_1}^{s_2}\hbar(s)\,ds\right|$$
$$\le \tfrac{25(s_2-s_1)}{576}\left(\left|\hbar'(s_1)\right|+\left|\hbar'(s_2)\right|\right),$$

which is the same result given in Remark 3 from [4].

Theorem 3.6. *Assuming all the assumptions of Theorem 3.2 are met, if $\left|\hbar^{(\rho)}\right|^{q}$ is generalized convex, then we have*

$$\left|\tfrac{1}{8^{\rho}}\left(\hbar(s_1)+3^{\rho}\hbar\left(\tfrac{2s_1+s_2}{3}\right)+3^{\rho}\hbar\left(\tfrac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\tfrac{\Gamma(\rho+1)}{(s_2-s_1)^{\rho}}\,{}_{s_1}I^{\rho}_{s_2}\hbar(s)\right|$$
$$\le \tfrac{(s_2-s_1)^{\rho}}{9^{\rho}}\left(\tfrac{\Gamma(1+p\rho)}{\Gamma(1+(p+1)\rho)}\right)^{\frac{1}{p}}\left(\tfrac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right)^{\frac{1}{q}}\left(\left(\tfrac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{\rho}{p}}\left(\left|\hbar^{(\rho)}(s_1)\right|^{q}+\left|\hbar^{(\rho)}\left(\tfrac{2s_1+s_2}{3}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\tfrac{1}{2}\right)^{\rho}\left(\left|\hbar^{(\rho)}\left(\tfrac{2s_1+s_2}{3}\right)\right|^{q}+\left|\hbar^{(\rho)}\left(\tfrac{s_1+2s_2}{3}\right)\right|^{q}\right)^{\frac{1}{q}}+\left(\tfrac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{\rho}{p}}\left(\left|\hbar^{(\rho)}\left(\tfrac{2s_1+s_2}{3}\right)\right|^{q}+\left|\hbar^{(\rho)}(s_2)\right|^{q}\right)^{\frac{1}{q}}\right),$$

where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 3.1 along with the generalized Hölder inequality, we get

$$\left|\tfrac{1}{8^{\rho}}\left(\hbar(s_1)+3^{\rho}\hbar\left(\tfrac{2s_1+s_2}{3}\right)+3^{\rho}\hbar\left(\tfrac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\tfrac{\Gamma(\rho+1)}{(s_2-s_1)^{\rho}}\,{}_{s_1}I^{\rho}_{s_2}\hbar(s)\right|$$
$$\le \tfrac{(s_2-s_1)^{\rho}}{9^{\rho}}\left(\left(\tfrac{1}{\Gamma(\rho+1)}\int_0^1\left|t-\tfrac{3}{8}\right|^{p\rho}(dt)^{\rho}\right)^{\frac{1}{p}}\left(\tfrac{1}{\Gamma(\rho+1)}\int_0^1\left|\hbar^{(\rho)}\left((1-t)s_1+t\tfrac{2s_1+s_2}{3}\right)\right|^{q}(dt)^{\rho}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|\frac{1}{2}-t\right|^{p\rho}(dt)^\rho\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|\hbar^{(\rho)}\left((1-t)\frac{2s_1+s_2}{3}+t\frac{s_1+2s_2}{3}\right)\right|^q(dt)^\rho\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|t-\frac{5}{8}\right|^{p\rho}(dt)^\rho\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|\hbar^{(\rho)}\left((1-t)\frac{s_1+2s_2}{3}+ts_2\right)\right|^q(dt)^\rho\right)^{\frac{1}{q}}\Bigg). \tag{3.10}$$

Since $\left|\hbar^{(\rho)}\right|^q$ is generalized convex, (3.10) gives

$$\left|\frac{1}{8^\rho}\left(\hbar(s_1)+3^\rho\hbar\left(\frac{2s_1+s_2}{3}\right)+3^\rho\hbar\left(\frac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\frac{\Gamma(\rho+1)}{(s_2-s_1)^\rho}\,{}_{s_1}I^\rho_{s_2}\hbar(s)\right|$$

$$\leq\frac{(s_2-s_1)^\rho}{9^\rho}\left(\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|t-\frac{3}{8}\right|^{p\rho}(dt)^\rho\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left((1-t)^\rho\left|\hbar^{(\rho)}(s_1)\right|^q+t^\rho\left|\hbar^{(\rho)}\left(\frac{2s_1+s_2}{3}\right)\right|^q\right)(dt)^\rho\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|\frac{1}{2}-t\right|^{p\rho}(dt)^\rho\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left((1-t)^\rho\left|\hbar^{(\rho)}\left(\frac{2s_1+s_2}{3}\right)\right|^q+t^\rho\left|\hbar^{(\rho)}\left(\frac{s_1+2s_2}{3}\right)\right|^q\right)(dt)^\rho\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|t-\frac{5}{8}\right|^{p\rho}(dt)^\rho\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left((1-t)^\rho\left|\hbar^{(\rho)}\left(\frac{s_1+2s_2}{3}\right)\right|^q+t^\rho\left|\hbar^{(\rho)}(s_2)\right|^q\right)(dt)^\rho\right)^{\frac{1}{q}}\right)$$

$$=\frac{(s_2-s_1)^\rho}{9^\rho}\left(\frac{\Gamma(1+p\rho)}{\Gamma(1+(p+1)\rho)}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right)^{\frac{1}{q}}\left(\left(\left(\frac{3}{8}\right)^{p+1}+\left(\frac{5}{8}\right)^{p+1}\right)^{\frac{\rho}{p}}\left(\left|\hbar^{(\rho)}(s_1)\right|^q+\left|\hbar^{(\rho)}\left(\frac{2s_1+s_2}{3}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{1}{2}\right)^\rho\left(\left|\hbar^{(\rho)}\left(\frac{2s_1+s_2}{3}\right)\right|^q+\left|\hbar^{(\rho)}\left(\frac{s_1+2s_2}{3}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left(\frac{5}{8}\right)^{p+1}+\left(\frac{3}{8}\right)^{p+1}\right)^{\frac{\rho}{p}}\left(\left|\hbar^{(\rho)}\left(\frac{2s_1+s_2}{3}\right)\right|^q+\left|\hbar^{(\rho)}(s_2)\right|^q\right)^{\frac{1}{q}}\right),$$

where we have used

$$\frac{1}{\Gamma(\rho+1)}\int_0^1\left|t-\frac{3}{8}\right|^{p\rho}(dt)^\rho=\frac{\Gamma(1+p\rho)}{\Gamma(1+(p+1)\rho)}\left(\left(\frac{3}{8}\right)^{p+1}+\left(\frac{5}{8}\right)^{p+1}\right)^\rho, \tag{3.11}$$

$$\frac{1}{\Gamma(\rho+1)}\int_0^1\left|\frac{1}{2}-t\right|^{p\rho}(dt)^\rho=\frac{\Gamma(1+p\rho)}{\Gamma(1+(p+1)\rho)}\left(\frac{1}{2}\right)^{p\rho}, \tag{3.12}$$

$$\frac{1}{\Gamma(\rho+1)}\int_0^1\left|t-\frac{5}{8}\right|^{p\rho}(dt)^\rho=\frac{\Gamma(1+p\rho)}{\Gamma(1+(p+1)\rho)}\left(\left(\frac{5}{8}\right)^{p+1}+\left(\frac{3}{8}\right)^{p+1}\right)^\rho \tag{3.13}$$

and

$$\frac{1}{\Gamma(\rho+1)}\int_0^1(1-t)^\rho(dt)^\rho=\frac{1}{\Gamma(\rho+1)}\int_0^1 t^\rho(dt)^\rho=\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}.$$

This concludes the demonstration. □

Corollary 3.7. *In Corollary 3.6, if $\rho\to 1$, we obtain*

$$\left|\frac{1}{8}\left(\hbar(s_1)+3\hbar\left(\frac{2s_1+s_2}{3}\right)+3\hbar\left(\frac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\frac{1}{s_2-s_1}\int_{s_1}^{s_2}\hbar(s)\,ds\right|$$

$$\leq \frac{s_2-s_1}{9}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(\frac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{1}{p}}\left(\frac{|\hbar'(s_1)|^q+\left|\hbar'\left(\frac{2s_1+s_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\frac{1}{2}\left(\frac{\left|\hbar'\left(\frac{2s_1+s_2}{3}\right)\right|^q+\left|\hbar'\left(\frac{s_1+2s_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{1}{p}}\left(\frac{\left|\hbar'\left(\frac{2s_1+s_2}{3}\right)\right|^q+|\hbar'(s_2)|^q}{2}\right)^{\frac{1}{q}}\right),$$

which is equivalent to the result given in Theorem 4 from [3].

Corollary 3.8. *In Theorem 3.6, using the generalized convexity of $\left|\hbar^{(\rho)}\right|$, we get*

$$\left|\frac{1}{8^\rho}\left(\hbar(s_1)+3^\rho\hbar\left(\frac{2s_1+s_2}{3}\right)+3^\rho\hbar\left(\frac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\frac{\Gamma(\rho+1)}{(s_2-s_1)^\rho}\,{}_{s_1}I^{\rho}_{s_2}\hbar(s)\right|$$

$$\leq \frac{(s_2-s_1)^\rho}{9^\rho}\left(\frac{\Gamma(1+p\rho)}{\Gamma(1+(p+1)\rho)}\right)^{\frac{1}{p}}\left(\frac{2^\rho\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right)^{\frac{1}{q}}\left(\left(\frac{1}{2}\right)^\rho\left(\frac{\left|\hbar^{(\rho)}(s_1)\right|+\left|\hbar^{(\rho)}(s_2)\right|}{2^\rho}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{\rho}{p}}\left(\left(\frac{5^\rho\left|\hbar^{(\rho)}(s_1)\right|+\left|\hbar^{(\rho)}(s_2)\right|}{6^\rho}\right)^{\frac{1}{q}}+\left(\frac{\left|\hbar^{(\rho)}(s_1)\right|^q+2^\rho\left|\hbar^{(\rho)}(s_2)\right|^q}{3^\rho}\right)^{\frac{1}{q}}\right)\right),$$

which is the same result obtained in Theorem 4 from [28].

Theorem 3.9. *Assuming all the assumptions of Theorem 3.2 are met, if $\left|\hbar^{(\rho)}\right|^q$ is generalized convex for $q>1$, then we have*

$$\left|\frac{1}{8^\rho}\left(\hbar(s_1)+3^\rho\hbar\left(\frac{2s_1+s_2}{3}\right)+3^\rho\hbar\left(\frac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\frac{\Gamma(\rho+1)}{(s_2-s_1)^\rho}\,{}_{s_1}I^{\rho}_{s_2}\hbar(s)\right|$$

$$\leq \frac{(s_2-s_1)^\rho}{9^\rho}\left(\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right)^{1-\frac{1}{q}}\left(\left(\frac{17}{32}\right)^{\rho\left(1-\frac{1}{q}\right)}\left(\left(\frac{131^\rho\Gamma(1+2\rho)}{256^\rho\Gamma(1+3\rho)}-\frac{35^\rho\Gamma(1+\rho)}{256^\rho\Gamma(1+2\rho)}\right)\left|\hbar^{(\rho)}(s_1)\right|^q\right.\right.$$

$$\left.+\left(\frac{229^\rho\Gamma(1+2\rho)}{256^\rho\Gamma(1+3\rho)}-\frac{69^\rho\Gamma(1+\rho)}{256^\rho\Gamma(1+2\rho)}\right)\left|\hbar^{(\rho)}\left(\frac{2s_1+s_2}{3}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{2}\right)^{\rho\left(1-\frac{1}{q}\right)}\left(\frac{3^\rho\Gamma(1+2\rho)}{4^\rho\Gamma(1+3\rho)}-\frac{\Gamma(1+\rho)}{4^\rho\Gamma(1+2\rho)}\right)^{\frac{1}{q}}\left(\left|\hbar^{(\rho)}\left(\frac{2s_1+s_2}{3}\right)\right|^q+\left|\hbar^{(\rho)}\left(\frac{s_1+2s_2}{3}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{17}{32}\right)^{\rho\left(1-\frac{1}{q}\right)}\left(\left(\frac{229^\rho\Gamma(1+2\rho)}{256^\rho\Gamma(1+3\rho)}-\frac{69^\rho\Gamma(1+\rho)}{256^\rho\Gamma(1+2\rho)}\right)\left|\hbar^{(\rho)}\left(\frac{s_1+2s_2}{3}\right)\right|^q\right.$$

$$\left.\left.+\left(\frac{131^\rho\Gamma(1+2\rho)}{256^\rho\Gamma(1+3\rho)}-\frac{35^\rho\Gamma(1+\rho)}{256^\rho\Gamma(1+2\rho)}\right)\left|\hbar^{(\rho)}(s_2)\right|^q\right)^{\frac{1}{q}}\right).$$

Proof. Using Lemma 3.1 along with the generalized power mean inequality, we get

$$\left|\frac{1}{8^\rho}\left(\hbar(s_1)+3^\rho\hbar\left(\frac{2s_1+s_2}{3}\right)+3^\rho\hbar\left(\frac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\frac{\Gamma(\rho+1)}{(s_2-s_1)^\rho}\,{}_{s_1}I^{\rho}_{s_2}\hbar(s)\right|$$

$$\leq \frac{(s_2-s_1)^\rho}{9^\rho}\left(\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|t-\frac{3}{8}\right|^\rho(dt)^\rho\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|t-\frac{3}{8}\right|^\rho\left|\hbar^{(\rho)}\left((1-t)s_1+t\frac{2s_1+s_2}{3}\right)\right|^q(dt)^\rho\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|\frac{1}{2}-t\right|^\rho(dt)^\rho\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|\frac{1}{2}-t\right|^\rho\left|\hbar^{(\rho)}\left((1-t)\frac{2s_1+s_2}{3}+t\frac{s_1+2s_2}{3}\right)\right|^q(dt)^\rho\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1 \left|t-\tfrac{5}{8}\right|^\rho (dt)^\rho\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1 \left|t-\tfrac{5}{8}\right|^\rho \left|\hbar^{(\rho)}\left((1-t)\tfrac{s_1+2s_2}{3}+ts_2\right)\right|^q (dt)^\rho\right)^{\frac{1}{q}}\Bigg). \tag{3.14}$$

Since $\left|\hbar^{(\rho)}\right|^q$ is generalized convex, (3.14) yields

$$\left|\tfrac{1}{8^\rho}\left(\hbar(s_1)+3^\rho\hbar\left(\tfrac{2s_1+s_2}{3}\right)+3^\rho\hbar\left(\tfrac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\tfrac{\Gamma(\rho+1)}{(s_2-s_1)^\rho}\,{}_{s_1}I_{s_2}^\rho\hbar(s)\right|$$

$$\leq\tfrac{(s_2-s_1)^\rho}{9^\rho}\Bigg(\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|t-\tfrac{3}{8}\right|^\rho(dt)^\rho\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|t-\tfrac{3}{8})\right|^\rho\left((1-t)^\rho\left|\hbar^{(\rho)}(s_1)\right|^q+t^\rho\left|\hbar^{(\rho)}\left(\tfrac{2s_1+s_2}{3}\right)\right|^q\right)(dt)^\rho\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|\tfrac{1}{2}-t\right|^\rho(dt)^\rho\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|\tfrac{1}{2}-t\right|^\rho\left((1-t)^\rho\left|\hbar^{(\rho)}\left(\tfrac{2s_1+s_2}{3}\right)\right|^q+t^\rho\left|\hbar^{(\rho)}\left(\tfrac{s_1+2s_2}{3}\right)\right|^q\right)(dt)^\rho\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|t-\tfrac{5}{8}\right|^\rho(dt)^\rho\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|t-\tfrac{5}{8}\right|^\rho\left((1-t)^\rho\left|\hbar^{(\rho)}\left(\tfrac{s_1+2s_2}{3}\right)\right|^q+t^\rho\left|\hbar^{(\rho)}(s_2)\right|^q\right)(dt)^\rho\right)^{\frac{1}{q}}\Bigg)$$

$$=\tfrac{(s_2-s_1)^\rho}{9^\rho}\Bigg(\left(\tfrac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\left(\tfrac{17}{32}\right)^\rho\right)^{1-\frac{1}{q}}\left(\frac{\left|\hbar^{(\rho)}(s_1)\right|^q}{\Gamma(\rho+1)}\int_0^1\left|t-\tfrac{3}{8}\right|^\rho(1-t)^\rho(dt)^\rho+\frac{\left|\hbar^{(\rho)}\left(\frac{2s_1+s_2}{3}\right)\right|^q}{\Gamma(\rho+1)}\int_0^1\left|t-\tfrac{3}{8}\right|^\rho t^\rho(dt)^\rho\right)^{\frac{1}{q}}$$

$$+\left(\tfrac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\left(\tfrac{1}{2}\right)^\rho\right)^{1-\frac{1}{q}}\left(\frac{\left|\hbar^{(\rho)}\left(\frac{2s_1+s_2}{3}\right)\right|^q}{\Gamma(\rho+1)}\int_0^1\left|\tfrac{1}{2}-t\right|^\rho(1-t)^\rho(dt)^\rho+\frac{\left|\hbar^{(\rho)}\left(\frac{s_1+2s_2}{3}\right)\right|^q}{\Gamma(\rho+1)}\int_0^1\left|\tfrac{1}{2}-t\right|^\rho t^\rho(dt)^\rho\right)^{\frac{1}{q}}$$

$$+\left(\tfrac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\left(\tfrac{17}{32}\right)^\rho\right)^{1-\frac{1}{q}}\left(\frac{\left|\hbar^{(\rho)}\left(\frac{s_1+2s_2}{3}\right)\right|^q}{\Gamma(\rho+1)}\int_0^1\left|t-\tfrac{5}{8}\right|^\rho(1-t)^\rho(dt)^\rho+\frac{\left|\hbar^{(\rho)}(s_2)\right|^q}{\Gamma(\rho+1)}\int_0^1\left|t-\tfrac{5}{8}\right|^\rho t^\rho(dt)^\rho\right)^{\frac{1}{q}}\Bigg)$$

$$=\tfrac{(s_2-s_1)^\rho}{9^\rho}\left(\tfrac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right)^{1-\frac{1}{q}}\Bigg(\left(\tfrac{17}{32}\right)^{\rho\left(1-\frac{1}{q}\right)}\left(\left(\tfrac{131^\rho\Gamma(1+2\rho)}{256^\rho\Gamma(1+3\rho)}-\tfrac{35^\rho\Gamma(1+\rho)}{256^\rho\Gamma(1+2\rho)}\right)\left|\hbar^{(\rho)}(s_1)\right|^q\right.$$

$$\left.+\left(\tfrac{229^\rho\Gamma(1+2\rho)}{256^\rho\Gamma(1+3\rho)}-\tfrac{69^\rho\Gamma(1+\rho)}{256^\rho\Gamma(1+2\rho)}\right)\left|\hbar^{(\rho)}\left(\tfrac{2s_1+s_2}{3}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\tfrac{1}{2}\right)^{\rho\left(1-\frac{1}{q}\right)}\left(\tfrac{3^\rho\Gamma(1+2\rho)}{4^\rho\Gamma(1+3\rho)}-\tfrac{\Gamma(1+\rho)}{4^\rho\Gamma(1+2\rho)}\right)^{\frac{1}{q}}\left(\left|\hbar^{(\rho)}\left(\tfrac{2s_1+s_2}{3}\right)\right|^q+\left|\hbar^{(\rho)}\left(\tfrac{s_1+2s_2}{3}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\tfrac{17}{32}\right)^{\rho\left(1-\frac{1}{q}\right)}\left(\left(\tfrac{229^\rho\Gamma(1+2\rho)}{256^\rho\Gamma(1+3\rho)}-\tfrac{69^\rho\Gamma(1+\rho)}{256^\rho\Gamma(1+2\rho)}\right)\left|\hbar^{(\rho)}\left(\tfrac{s_1+2s_2}{3}\right)\right|^q\right.$$

$$\left.+\left(\tfrac{131^\rho\Gamma(1+2\rho)}{256^\rho\Gamma(1+3\rho)}-\tfrac{35^\rho\Gamma(1+\rho)}{256^\rho\Gamma(1+2\rho)}\right)\left|\hbar^{(\rho)}(s_2)\right|^q\right)^{\frac{1}{q}}\Bigg),$$

where we have used (3.7)-(3.9) and

$$\frac{1}{\Gamma(\rho+1)}\int_0^1\left|t-\tfrac{3}{8}\right|^\rho(dt)^\rho=\tfrac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\left(\tfrac{17}{32}\right)^\rho, \tag{3.15}$$

$$\frac{1}{\Gamma(\rho+1)}\int_0^1 \left|\tfrac{1}{2}-t\right|^{\rho}(dt)^{\rho}=\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\left(\tfrac{1}{2}\right)^{\rho} \tag{3.16}$$

and

$$\frac{1}{\Gamma(\rho+1)}\int_0^1 \left|t-\tfrac{5}{8}\right|^{\rho}(dt)^{\rho}=\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\left(\tfrac{17}{32}\right)^{\rho}. \tag{3.17}$$

This concludes the demonstration. □

Corollary 3.10. *In Theorem 3.9, if $\rho\to 1$, we obtain*

$$\begin{aligned}
&\left|\tfrac{1}{8}\left(\hbar(s_1)+3\hbar\left(\tfrac{2s_1+s_2}{3}\right)+3\hbar\left(\tfrac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\tfrac{1}{s_2-s_1}\int_{s_1}^{s_2}\hbar(s)\,ds\right|\\
\le&\tfrac{(s_2-s_1)^{\rho}}{9^{\rho}}\left(\tfrac{1}{2}\right)^{1-\frac{1}{q}}\left(\left(\tfrac{17}{32}\right)^{1-\frac{1}{q}}\left(\tfrac{157}{1536}\left|\hbar^{(\rho)}(s_1)\right|^q+\tfrac{251}{1536}\left|\hbar^{(\rho)}\left(\tfrac{2s_1+s_2}{3}\right)\right|^q\right)^{\frac{1}{q}}\right.\\
&+\left(\tfrac{1}{2}\right)^{1-\frac{1}{q}}\left(\tfrac{1}{8}\right)^{\frac{1}{q}}\left(\left|\hbar^{(\rho)}\left(\tfrac{2s_1+s_2}{3}\right)\right|^q+\left|\hbar^{(\rho)}\left(\tfrac{s_1+2s_2}{3}\right)\right|^q\right)^{\frac{1}{q}}\\
&\left.+\left(\tfrac{17}{32}\right)^{1-\frac{1}{q}}\left(\tfrac{251}{1536}\left|\hbar^{(\rho)}\left(\tfrac{s_1+2s_2}{3}\right)\right|^q+\tfrac{157}{1536}\left|\hbar^{(\rho)}(s_2)\right|^q\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Corollary 3.11. *In Theorem 3.9, using the generalized convexity of $\left|\hbar^{(\rho)}\right|^q$, we get*

$$\begin{aligned}
&\left|\tfrac{1}{8^{\rho}}\left(\hbar(s_1)+3^{\rho}\hbar\left(\tfrac{2s_1+s_2}{3}\right)+3^{\rho}\hbar\left(\tfrac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\tfrac{\Gamma(\rho+1)}{(s_2-s_1)^{\rho}}\,{}_{s_1}I^{\rho}_{s_2}\hbar(s)\right|\\
\le&\tfrac{(s_2-s_1)^{\rho}}{9^{\rho}}\left(\tfrac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right)^{1-\frac{1}{q}}\left(\left(\tfrac{17}{32}\right)^{\rho\left(1-\frac{1}{q}\right)}\right.\\
&\times\left(\left(\tfrac{851^{\rho}\Gamma(1+2\rho)}{768^{\rho}\Gamma(1+3\rho)}-\tfrac{243^{\rho}\Gamma(1+\rho)}{768^{\rho}\Gamma(1+2\rho)}\right)\left|\hbar^{(\rho)}(s_1)\right|^q+\left(\tfrac{229^{\rho}\Gamma(1+2\rho)}{768^{\rho}\Gamma(1+3\rho)}-\tfrac{69^{\rho}\Gamma(1+\rho)}{768^{\rho}\Gamma(1+2\rho)}\right)\left|\hbar^{(\rho)}(s_2)\right|^q\right)^{\frac{1}{q}}\\
&+\left(\tfrac{1}{2}\right)^{\rho\left(1-\frac{1}{q}\right)}\left(\tfrac{3^{\rho}\Gamma(1+2\rho)}{4^{\rho}\Gamma(1+3\rho)}-\tfrac{\Gamma(1+\rho)}{4^{\rho}\Gamma(1+2\rho)}\right)^{\frac{1}{q}}\left(\left|\hbar^{(\rho)}(s_1)\right|^q+\left|\hbar^{(\rho)}(s_2)\right|^q\right)^{\frac{1}{q}}+\left(\tfrac{17}{32}\right)^{\rho\left(1-\frac{1}{q}\right)}\\
&\left.\times\left(\left(\tfrac{229^{\rho}\Gamma(1+2\rho)}{768^{\rho}\Gamma(1+3\rho)}-\tfrac{69^{\rho}\Gamma(1+\rho)}{768^{\rho}\Gamma(1+2\rho)}\right)\left|\hbar^{(\rho)}(s_1)\right|^q+\left(\tfrac{851^{\rho}\Gamma(1+2\rho)}{768^{\rho}\Gamma(1+3\rho)}-\tfrac{243^{\rho}\Gamma(1+\rho)}{768^{\rho}\Gamma(1+2\rho)}\right)\left|\hbar^{(\rho)}(s_2)\right|^q\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Theorem 3.12. *Assuming all the assumptions of Theorem 3.2 are met, if $\left|\hbar^{(\rho)}\right|^q$ is generalized convex for $q>1$, then we have*

$$\begin{aligned}
&\left|\tfrac{1}{8^{\rho}}\left(\hbar(s_1)+3^{\rho}\hbar\left(\tfrac{2s_1+s_2}{3}\right)+3^{\rho}\hbar\left(\tfrac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\tfrac{\Gamma(\rho+1)}{(s_2-s_1)^{\rho}}\,{}_{s_1}I^{\rho}_{s_2}\hbar(s)\right|\\
\le&\tfrac{(s_2-s_1)^{\rho}}{9^{\rho}}\left(\left(\mathscr{U}_1(\rho)\right)^{1-\frac{1}{q}}\left(\mathscr{U}_2(\rho)\left|\hbar^{(\rho)}(s_1)\right|^q+\mathscr{U}_3(\rho)\left|\hbar^{(\rho)}\left(\tfrac{2s_1+s_2}{3}\right)\right|^q\right)^{\frac{1}{q}}\right.\\
&+\left(\mathscr{U}_4(\rho)\right)^{1-\frac{1}{q}}\left(\mathscr{U}_3(\rho)\left|\hbar^{(\rho)}(s_1)\right|^q+\mathscr{U}_5(\rho)\left|\hbar^{(\rho)}\left(\tfrac{2s_1+s_2}{3}\right)\right|^q\right)^{\frac{1}{q}}\\
&+\left(\mathscr{V}_1(\rho)\right)^{1-\frac{1}{q}}\left(\mathscr{V}_2(\rho)\left|\hbar^{(\rho)}\left(\tfrac{2s_1+s_2}{3}\right)\right|^q+\mathscr{V}_3(\rho)\left|\hbar^{(\rho)}\left(\tfrac{s_1+2s_2}{3}\right)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\mathscr{V}_4(\rho)\right)^{1-\frac{1}{q}}\left(\mathscr{V}_3(\rho)\left|\hbar^{(\rho)}\left(\tfrac{2s_1+s_2}{3}\right)\right|^q+\mathscr{V}_5(\rho)\left|\hbar^{(\rho)}\left(\tfrac{s_1+2s_2}{3}\right)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\mathscr{W}_1(\rho)\right)^{1-\frac{1}{q}}\left(\mathscr{W}_2(\rho)\left|\hbar^{(\rho)}\left(\tfrac{s_1+2s_2}{3}\right)\right|^q+\mathscr{W}_3(\rho)\left|\hbar^{(\rho)}(s_2)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\mathscr{W}_4(\rho)\right)^{1-\frac{1}{q}}\left(\mathscr{W}_3(\rho)\left|\hbar^{(\rho)}\left(\tfrac{s_1+2s_2}{3}\right)\right|^q+\mathscr{W}_5(\rho)\left|\hbar^{(\rho)}(s_2)\right|^q\right)^{\frac{1}{q}}\Bigg),$$

where $\mathscr{U}_i, \mathscr{V}_i$ and $\mathscr{W}_i$ are defined for $i=1,2,...,5$ by (3.18)-(3.32), respectively.

Proof. Using Lemma 3.1, the improved generalized power mean inequality along with the generalized convexity of $\left|\hbar^{(\rho)}\right|^q$, we have

$$\left|\frac{1}{8^\rho}\left(\hbar(s_1)+3^\rho\hbar\left(\tfrac{2s_1+s_2}{3}\right)+3^\rho\hbar\left(\tfrac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\frac{\Gamma(\rho+1)}{(s_2-s_1)^\rho}\,{}_{s_1}I^{\rho}_{s_2}\hbar(s)\right|$$
$$\le\frac{(s_2-s_1)^\rho}{9^\rho}\Bigg(\left(\frac{1}{\Gamma(\rho+1)}\int_0^1(1-t)^\rho\left|t-\tfrac{3}{8}\right|^\rho(dt)^\rho\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1(1-t)^\rho\left|t-\tfrac{3}{8}\right|^\rho\left|\hbar^{(\rho)}\left((1-t)s_1+t\tfrac{2s_1+s_2}{3}\right)\right|^q(dt)^\rho\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1t^\rho\left|t-\tfrac{3}{8}\right|^\rho(dt)^\rho\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1t^\rho\left|t-\tfrac{3}{8}\right|^\rho\left|\hbar^{(\rho)}\left((1-t)s_1+t\tfrac{2s_1+s_2}{3}\right)\right|^q(dt)^\rho\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1(1-t)^\rho\left|\tfrac{1}{2}-t\right|^\rho(dt)^\rho\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1(1-t)^\rho\left|\tfrac{1}{2}-t\right|^\rho\left|\hbar^{(\rho)}\left((1-t)\tfrac{2s_1+s_2}{3}+t\tfrac{s_1+2s_2}{3}\right)\right|^q(dt)^\rho\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1t^\rho\left|\tfrac{1}{2}-t\right|^\rho(dt)^\rho\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1t^\rho\left|\tfrac{1}{2}-t\right|^\rho\left|\hbar^{(\rho)}\left((1-t)\tfrac{2s_1+s_2}{3}+t\tfrac{s_1+2s_2}{3}\right)\right|^q(dt)^\rho\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1(1-t)^\rho\left|t-\tfrac{5}{8}\right|^\rho(dt)^\rho\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1(1-t)^\rho\left|t-\tfrac{5}{8}\right|^\rho\left|\hbar^{(\rho)}\left((1-t)\tfrac{s_1+2s_2}{3}+ts_2\right)\right|^q(dt)^\rho\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1t^\rho\left|t-\tfrac{5}{8}\right|^\rho(dt)^\rho\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1t^\rho\left|t-\tfrac{5}{8}\right|^\rho\left|\hbar^{(\rho)}\left((1-t)\tfrac{s_1+2s_2}{3}+ts_2\right)\right|^q(dt)^\rho\right)^{\frac{1}{q}}\Bigg)$$
$$\le\frac{(s_2-s_1)^\rho}{9^\rho}\Bigg(\left(\frac{1}{\Gamma(\rho+1)}\int_0^1(1-t)^\rho\left|t-\tfrac{3}{8}\right|^\rho(dt)^\rho\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{1}{\Gamma(\rho+1)}\int_0^1(1-t)^\rho\left|t-\tfrac{3}{8}\right|^\rho\left((1-t)^\rho\left|\hbar^{(\rho)}(s_1)\right|^q+t^\rho\left|\hbar^{(\rho)}\left(\tfrac{2s_1+s_2}{3}\right)\right|^q\right)(dt)^\rho\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1t^\rho\left|t-\tfrac{3}{8}\right|^\rho(dt)^\rho\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{1}{\Gamma(\rho+1)}\int_0^1 t^\rho\left|t-\frac{3}{8}\right|^\rho\left((1-t)^\rho\left|\hbar^{(\rho)}(s_1)\right|^q+t^\rho\left|\hbar^{(\rho)}\left(\frac{2s_1+s_2}{3}\right)\right|^q\right)(dt)^\rho\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1 (1-t)^\rho\left|\frac{1}{2}-t\right|^\rho(dt)^\rho\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{1}{\Gamma(\rho+1)}\int_0^1 (1-t)^\rho\left|\frac{1}{2}-t\right|^\rho\left((1-t)^\rho\left|\hbar^{(\rho)}\left(\frac{2s_1+s_2}{3}\right)\right|^q+t^\rho\left|\hbar^{(\rho)}\left(\frac{s_1+2s_2}{3}\right)\right|^q\right)(dt)^\rho\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1 t^\rho\left|\frac{1}{2}-t\right|^\rho(dt)^\rho\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{1}{\Gamma(\rho+1)}\int_0^1 t^\rho\left|\frac{1}{2}-t\right|^\rho\left((1-t)^\rho\left|\hbar^{(\rho)}\left(\frac{2s_1+s_2}{3}\right)\right|^q+t^\rho\left|\hbar^{(\rho)}\left(\frac{s_1+2s_2}{3}\right)\right|^q\right)(dt)^\rho\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1 (1-t)^\rho\left|t-\frac{5}{8}\right|^\rho(dt)^\rho\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{1}{\Gamma(\rho+1)}\int_0^1 (1-t)^\rho\left|t-\frac{5}{8}\right|^\rho\left((1-t)^\rho\left|\hbar^{(\rho)}\left(\frac{s_1+2s_2}{3}\right)\right|^q+t^\rho\left|\hbar^{(\rho)}(s_2)\right|^q\right)(dt)^\rho\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1 t^\rho\left|t-\frac{5}{8}\right|^\rho(dt)^\rho\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\frac{1}{\Gamma(\rho+1)}\int_0^1 t^\rho\left|t-\frac{5}{8}\right|^\rho\left((1-t)^\rho\left|\hbar^{(\rho)}\left(\frac{s_1+2s_2}{3}\right)\right|^q+t^\rho\left|\hbar^{(\rho)}(s_2)\right|^q\right)(dt)^\rho\right)^{\frac{1}{q}}\right)$$

$$=\frac{(s_2-s_1)^\rho}{9^\rho}\left(\left(\mathscr{U}_1(\rho)\right)^{1-\frac{1}{q}}\left(\mathscr{U}_2(\rho)\left|\hbar^{(\rho)}(s_1)\right|^q+\mathscr{U}_3(\rho)\left|\hbar^{(\rho)}\left(\frac{2s_1+s_2}{3}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$+\left(\mathscr{U}_4(\rho)\right)^{1-\frac{1}{q}}\left(\mathscr{U}_3(\rho)\left|\hbar^{(\rho)}(s_1)\right|^q+\mathscr{U}_5(\rho)\left|\hbar^{(\rho)}\left(\frac{2s_1+s_2}{3}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\mathscr{V}_1(\rho)\right)^{1-\frac{1}{q}}\left(\mathscr{V}_2(\rho)\left|\hbar^{(\rho)}\left(\frac{2s_1+s_2}{3}\right)\right|^q+\mathscr{V}_3(\rho)\left|\hbar^{(\rho)}\left(\frac{s_1+2s_2}{3}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\mathscr{V}_4(\rho)\right)^{1-\frac{1}{q}}\left(\mathscr{V}_3(\rho)\left|\hbar^{(\rho)}\left(\frac{2s_1+s_2}{3}\right)\right|^q+\mathscr{V}_5(\rho)\left|\hbar^{(\rho)}\left(\frac{s_1+2s_2}{3}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\mathscr{W}_1(\rho)\right)^{1-\frac{1}{q}}\left(\mathscr{W}_2(\rho)\left|\hbar^{(\rho)}\left(\frac{s_1+2s_2}{3}\right)\right|^q+\mathscr{W}_3(\rho)\left|\hbar^{(\rho)}(s_2)\right|^q\right)^{\frac{1}{q}}$$

$$\left.+\left(\mathscr{W}_4(\rho)\right)^{1-\frac{1}{q}}\left(\mathscr{W}_3(\rho)\left|\hbar^{(\rho)}\left(\frac{s_1+2s_2}{3}\right)\right|^q+\mathscr{W}_5(\rho)\left|\hbar^{(\rho)}(s_2)\right|^q\right)^{\frac{1}{q}}\right),$$

where we have used

$$\mathscr{U}_1(\rho)=\frac{1}{\Gamma(\rho+1)}\int_0^1(1-t)^\rho\left|t-\tfrac{3}{8}\right|^\rho(dt)^\rho=\left(\tfrac{131}{256}\right)^\rho\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}-\left(\tfrac{35}{256}\right)^\rho\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}, \tag{3.18}$$

$$\mathscr{U}_2(\rho)=\frac{1}{\Gamma(\rho+1)}\int_0^1(1-t)^{2\rho}\left|t-\tfrac{3}{8}\right|^\rho(dt)^\rho=\left(\tfrac{1423}{2048}\right)^\rho\frac{\Gamma(1+3\rho)}{\Gamma(1+4\rho)}-\left(\tfrac{655}{2048}\right)^\rho\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}, \tag{3.19}$$

$$\begin{aligned}\mathscr{U}_3(\rho)&=\frac{1}{\Gamma(\rho+1)}\int_0^1(1-t)^\rho t^\rho\left|t-\tfrac{3}{8}\right|^\rho(dt)^\rho \\ &=-\left(\tfrac{1967}{2048}\right)^\rho\frac{\Gamma(1+3\rho)}{\Gamma(1+4\rho)}+\left(\tfrac{2519}{2048}\right)^\rho\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}-\left(\tfrac{69}{256}\right)^\rho\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)},\end{aligned} \tag{3.20}$$

$$\mathscr{U}_4(\rho)=\frac{1}{\Gamma(\rho+1)}\int_0^1 t^\rho\left|t-\tfrac{3}{8}\right|^\rho(dt)^\rho=\left(\tfrac{229}{256}\right)^\rho\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}-\left(\tfrac{69}{256}\right)^\rho\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}, \tag{3.21}$$

$$\mathscr{U}_5(\rho)=\frac{1}{\Gamma(\rho+1)}\int_0^1 t^{2\rho}\left|t-\tfrac{3}{8}\right|^\rho(dt)^\rho=\left(\tfrac{1967}{2048}\right)^\rho\frac{\Gamma(1+3\rho)}{\Gamma(1+4\rho)}-\left(\tfrac{687}{2048}\right)^\rho\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}, \tag{3.22}$$

$$\mathscr{V}_1(\rho)=\frac{1}{\Gamma(\rho+1)}\int_0^1(1-t)^\rho\left|\tfrac{1}{2}-t\right|^\rho(dt)^\rho=\left(\tfrac{3}{4}\right)^\rho\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}-\left(\tfrac{1}{4}\right)^\rho\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}, \tag{3.23}$$

$$\mathscr{V}_2(\rho)=\frac{1}{\Gamma(\rho+1)}\int_0^1(1-t)^{2\rho}\left|\tfrac{1}{2}-t\right|^\rho(dt)^\rho=\left(\tfrac{7}{8}\right)^\rho\frac{\Gamma(1+3\rho)}{\Gamma(1+4\rho)}-\left(\tfrac{3}{8}\right)^\rho\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}, \tag{3.24}$$

$$\begin{aligned}\mathscr{V}_3(\rho)&=\frac{1}{\Gamma(\rho+1)}\int_0^1 t^\rho(1-t)^\rho\left|\tfrac{1}{2}-t\right|^\rho(dt)^\rho \\ &=\left(\tfrac{3}{8}\right)^\rho\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}-\left(\tfrac{1}{4}\right)^\rho\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}-\left(\tfrac{1}{8}\right)^\rho\frac{\Gamma(1+3\rho)}{\Gamma(1+4\rho)},\end{aligned} \tag{3.25}$$

$$\mathscr{V}_4(\rho)=\frac{1}{\Gamma(\rho+1)}\int_0^1 t^\rho\left|\tfrac{1}{2}-t\right|^\rho(dt)^\rho=\left(\tfrac{3}{4}\right)^\rho\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}-\left(\tfrac{1}{4}\right)^\rho\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}, \tag{3.26}$$

$$\mathscr{V}_5(\rho)=\frac{1}{\Gamma(\rho+1)}\int_0^1 t^{2\rho}\left|\tfrac{1}{2}-t\right|^\rho(dt)^\rho=\left(\tfrac{7}{8}\right)^\rho\frac{\Gamma(1+3\rho)}{\Gamma(1+4\rho)}-\left(\tfrac{3}{8}\right)^\rho\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}, \tag{3.27}$$

$$\mathscr{W}_1(\rho)=\frac{1}{\Gamma(\rho+1)}\int_0^1(1-t)^{\rho}\left|t-\tfrac{5}{8}\right|^{\rho}(dt)^{\rho}=\left(\tfrac{229}{256}\right)^{\rho}\tfrac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}-\left(\tfrac{69}{256}\right)^{\rho}\tfrac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}, \tag{3.28}$$

$$\mathscr{W}_2(\rho)=\frac{1}{\Gamma(\rho+1)}\int_0^1(1-t)^{2\rho}\left|t-\tfrac{5}{8}\right|^{\rho}(dt)^{\rho}=\left(\tfrac{1967}{2048}\right)^{\rho}\tfrac{\Gamma(1+3\rho)}{\Gamma(1+4\rho)}-\left(\tfrac{687}{2048}\right)^{\rho}\tfrac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}, \tag{3.29}$$

$$\begin{aligned}\mathscr{W}_3(\rho)&=\frac{1}{\Gamma(\rho+1)}\int_0^1 t^{\rho}(1-t)^{\rho}\left|t-\tfrac{5}{8}\right|^{\rho}(dt)^{\rho} \\ &=\left(\tfrac{1703}{2048}\right)^{\rho}\tfrac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}-\left(\tfrac{1423}{2048}\right)^{\rho}\tfrac{\Gamma(1+3\rho)}{\Gamma(1+4\rho)}-\left(\tfrac{35}{256}\right)^{\rho}\tfrac{\Gamma(1+\rho)}{\Gamma(1+2\rho)},\end{aligned} \tag{3.30}$$

$$\mathscr{W}_4(\rho)=\frac{1}{\Gamma(\rho+1)}\int_0^1 t^{\rho}\left|t-\tfrac{5}{8}\right|^{\rho}(dt)^{\rho}=\left(\tfrac{131}{256}\right)^{\rho}\tfrac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}-\left(\tfrac{35}{256}\right)^{\rho}\tfrac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}, \tag{3.31}$$

$$\mathscr{W}_5(\rho)=\frac{1}{\Gamma(\rho+1)}\int_0^1 t^{2\rho}\left|t-\tfrac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right|^{\rho}(dt)^{\rho}=\left(\tfrac{1423}{2048}\right)^{\rho}\tfrac{\Gamma(1+3\rho)}{\Gamma(1+4\rho)}-\left(\tfrac{655}{2048}\right)^{\rho}\tfrac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}. \tag{3.32}$$

This concludes the demonstration. □

Corollary 3.13. *In Theorem 3.12, using the generalized convexity of $\left|\hbar^{(\rho)}\right|^q$, we get*

$$\begin{aligned}&\left|\tfrac{1}{8^{\rho}}\left(\hbar(s_1)+3^{\rho}\hbar\left(\tfrac{2s_1+s_2}{3}\right)+3^{\rho}\hbar\left(\tfrac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\tfrac{\Gamma(\rho+1)}{(s_2-s_1)^{\rho}}\,{}_{s_1}I_{s_2}^{\rho}\hbar(s)\right| \\
&\leq\tfrac{(s_2-s_1)^{\rho}}{9^{\rho}}\left((\mathscr{U}_1(\rho))^{1-\frac{1}{q}}\left(\tfrac{3^{\rho}\mathscr{U}_2(\rho)+2^{\rho}\mathscr{U}_3(\rho)}{3^{\rho}}\left|\hbar^{(\rho)}(s_1)\right|^q+\tfrac{\mathscr{U}_3(\rho)}{3^{\rho}}\left|\hbar^{(\rho)}(s_2)\right|^q\right)^{\frac{1}{q}}\right. \\
&+(\mathscr{U}_4(\rho))^{1-\frac{1}{q}}\left(\tfrac{3^{\rho}\mathscr{U}_3(\rho)+2^{\rho}\mathscr{U}_5(\rho)}{3^{\rho}}\left|\hbar^{(\rho)}(s_1)\right|^q+\tfrac{\mathscr{U}_5(\rho)}{3^{\rho}}\left|\hbar^{(\rho)}(s_2)\right|^q\right)^{\frac{1}{q}} \\
&+(\mathscr{V}_1(\rho))^{1-\frac{1}{q}}\left(\tfrac{2^{\rho}\mathscr{V}_2(\rho)+\mathscr{V}_3(\rho)}{3^{\rho}}\left|\hbar^{(\rho)}(s_1)\right|^q+\tfrac{\mathscr{V}_2(\rho)+2^{\rho}\mathscr{V}_3(\rho)}{3^{\rho}}\left|\hbar^{(\rho)}(s_2)\right|^q\right)^{\frac{1}{q}} \\
&+(\mathscr{V}_4(\rho))^{1-\frac{1}{q}}\left(\tfrac{2^{\rho}\mathscr{V}_3(\rho)+\mathscr{V}_5(\rho)}{3^{\rho}}\left|\hbar^{(\rho)}(s_1)\right|^q+\tfrac{\mathscr{V}_3(\rho)+2^{\rho}\mathscr{V}_5(\rho)}{3^{\rho}}\left|\hbar^{(\rho)}(s_2)\right|^q\right)^{\frac{1}{q}} \\
&+(\mathscr{W}_1(\rho))^{1-\frac{1}{q}}\left(\tfrac{\mathscr{W}_2(\rho)}{3^{\rho}}\left|\hbar^{(\rho)}(s_1)\right|^q+\tfrac{2^{\rho}\mathscr{W}_2(\rho)+3^{\rho}\mathscr{W}_3(\rho)}{3^{\rho}}\left|\hbar^{(\rho)}(s_2)\right|^q\right)^{\frac{1}{q}} \\
&\left.+(\mathscr{W}_4(\rho))^{1-\frac{1}{q}}\left(\tfrac{\mathscr{W}_3(\rho)}{3^{\rho}}\left|\hbar^{(\rho)}(s_1)\right|^q+\tfrac{2^{\rho}\mathscr{W}_3(\rho)+3^{\rho}\mathscr{W}_5(\rho)}{3^{\rho}}\left|\hbar^{(\rho)}(s_2)\right|^q\right)^{\frac{1}{q}}\right).\end{aligned}$$

Theorem 3.14. *Assuming all the assumptions of Theorem 3.2 are met, if $\left|\hbar^{(\rho)}\right|^q$ is generalized concave for $q>1$, then we have*

$$\begin{aligned}&\left|\tfrac{1}{8^{\rho}}\left(\hbar(s_1)+3^{\rho}\hbar\left(\tfrac{2s_1+s_2}{3}\right)+3^{\rho}\hbar\left(\tfrac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\tfrac{\Gamma(\rho+1)}{(s_2-s_1)^{\rho}}\,{}_{s_1}I_{s_2}^{\rho}\hbar(s)\right| \\
&\leq\tfrac{(s_2-s_1)^{\rho}}{9^{\rho}}\left(\tfrac{(s_2-s_1)^{\rho}}{3^{\rho}\Gamma(\rho+1)}\right)^{\frac{1}{q}}\left(\tfrac{\Gamma(1+p\rho)}{\Gamma(1+(p+1)\rho)}\right)^{\frac{1}{p}}\left(\left(\tfrac{1}{2}\right)^{\frac{\rho}{p}}\left|\hbar^{(\rho)}\left(\tfrac{s_1+s_2}{2}\right)\right|\right.\end{aligned}$$

$$+\left(\frac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{\rho}{p}}\left(\left|\hbar^{(\rho)}\left(\frac{5s_1+s_2}{6}\right)\right|+\left|\hbar^{(\rho)}\left(\frac{s_1+5s_2}{6}\right)\right|\right)\Bigg),$$

where $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 3.1, the generalized Hölder inequality along with the generalized concavity of $\left|\hbar^{(\rho)}\right|^q$, we have

$$\left|\frac{1}{8^\rho}\left(\hbar(s_1)+3^\rho\hbar\left(\frac{2s_1+s_2}{3}\right)+3^\rho\hbar\left(\frac{s_1+2s_2}{3}\right)+\hbar(s_2)\right)-\frac{\Gamma(\rho+1)}{(s_2-s_1)^\rho}\,{}_{s_1}I^\rho_{s_2}\hbar(s)\right|$$

$$\leq\frac{(s_2-s_1)^\rho}{9^\rho}\left(\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|t-\frac{3}{8}\right|^{p\rho}(dt)^\rho\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|\hbar^{(\rho)}\left((1-t)s_1+t\frac{2s_1+s_2}{3}\right)\right|^q(dt)^\rho\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|\frac{1}{2}-t\right|^{p\rho}(dt)^\rho\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|\hbar^{(\rho)}\left((1-t)\frac{2s_1+s_2}{3}+t\frac{s_1+2s_2}{3}\right)\right|^q(dt)^\rho\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|t-\frac{5}{8}\right|^{p\rho}(dt)^\rho\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\rho+1)}\int_0^1\left|\hbar^{(\rho)}\left((1-t)\frac{s_1+2s_2}{3}+ts_2\right)\right|^q(dt)^\rho\right)^{\frac{1}{q}}\right)$$

$$\leq\frac{(s_2-s_1)^\rho}{9^\rho}\Bigg(\left(\frac{\Gamma(1+p\rho)}{\Gamma(1+(p+1)\rho)}\left(\left(\frac{3}{8}\right)^{p+1}+\left(\frac{5}{8}\right)^{p+1}\right)^\rho\right)^{\frac{1}{p}}\left(\frac{(s_2-s_1)^\rho}{3^\rho\Gamma(\rho+1)}\right)^{\frac{1}{q}}\left|\hbar^{(\rho)}\left(\frac{5s_1+s_2}{6}\right)\right|$$

$$+\left(\frac{\Gamma(1+p\rho)}{\Gamma(1+(p+1)\rho)}\left(\frac{1}{2}\right)^\rho\right)^{\frac{1}{p}}\left(\frac{(s_2-s_1)^\rho}{3^\rho\Gamma(\rho+1)}\right)^{\frac{1}{q}}\left|\hbar^{(\rho)}\left(\frac{s_1+s_2}{2}\right)\right|$$

$$+\left(\frac{\Gamma(1+p\rho)}{\Gamma(1+(p+1)\rho)}\left(\left(\frac{5}{8}\right)^{p+1}+\left(\frac{3}{8}\right)^{p+1}\right)^\rho\right)^{\frac{1}{p}}\left(\frac{(s_2-s_1)^\rho}{3^\rho\Gamma(\rho+1)}\right)^{\frac{1}{q}}\left|\hbar^{(\rho)}\left(\frac{s_1+5s_2}{6}\right)\right|\Bigg)$$

$$=\frac{(s_2-s_1)^\rho}{9^\rho}\left(\frac{(s_2-s_1)^\rho}{3^\rho\Gamma(\rho+1)}\right)^{\frac{1}{q}}\left(\frac{\Gamma(1+p\rho)}{\Gamma(1+(p+1)\rho)}\right)^{\frac{1}{p}}\Bigg(\left(\frac{1}{2}\right)^{\frac{\rho}{p}}\left|\hbar^{(\rho)}\left(\frac{s_1+s_2}{2}\right)\right|$$

$$+\left(\frac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{\rho}{p}}\left(\left|\hbar^{(\rho)}\left(\frac{5s_1+s_2}{6}\right)\right|+\left|\hbar^{(\rho)}\left(\frac{s_1+5s_2}{6}\right)\right|\right)\Bigg),$$

where we have used (3.11)-(3.13). The proof is completed. □

4. Application to special means

For any arbitrary real numbers s_1 and s_2:

The generalized arithmetic mean is given by: $A(s_1,s_2)=\frac{s_1^\rho+s_2^\rho}{2^\rho}$.

The generalized p-Logarithmic mean is defined as: $L_p(s_1,s_2)=\left[\frac{\Gamma(1+p\rho)}{\Gamma(1+(p+1)\rho)}\left(\frac{s_2^{(p+1)\rho}-s_1^{(p+1)\rho}}{(s_2-s_1)^\rho}\right)\right]^{\frac{1}{p}}$, $s_1,s_2\in\mathbb{R},s_1\neq s_2$ and $p\in\mathbb{Z}\backslash\{-1,0\}$.

Proposition 4.1. *Let* $s_1,s_2\in\mathbb{R}$ *with* $0<s_1<s_2,q>1$ *and* $n\geq2$, *then we have*

$$\left|\frac{2^\rho A(s_1^n,s_2^n)+3^\rho A^3(s_1,s_1,s_2)+3^\rho A^3(s_1,s_2,s_2)}{8^\rho}-\Gamma(\rho+1)L_n^n(s_1,s_2)\right|$$

$$\leq\frac{(s_2-s_1)^\rho}{9^\rho}\left(\frac{\Gamma(1+p\rho)}{\Gamma(1+(p+1)\rho)}\right)^{\frac{1}{p}}\left(\frac{2^\rho\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right)^{\frac{1}{q}}\left(\frac{\Gamma(1+n\rho)}{\Gamma(1+(n-1)\rho)}\right)^{\frac{1}{q}}\left(\left(\frac{1}{2}\right)^\rho\left(\frac{s_1^{q(n-1)\rho}+s_2^{q(n-1)\rho}}{2^\rho}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{\rho}{p}}\left(\left(\frac{5^{\rho}s_1^{q(n-1)\rho}+s_2^{q(n-1)\rho}}{6^{\rho}}\right)^{\frac{1}{q}}+\left(\frac{s_1^{q(n-1)\rho}+2^{\rho}s_2^{q(n-1)\rho}}{3^{\rho}}\right)^{\frac{1}{q}}\right)\Bigg).$$

Proof. This follows from Corollary 3.8, applied to the function $\hbar(s)=x^{n\rho}$ where $\hbar:(0,+\infty)\to\mathbb{R}^{\rho}$. □

5. Conclusion

In this study, inspired by prior research, notably [28], we have introduced a novel identity involving local fractional integrals and utilized it to establish a series of Newton-type inequalities for functions with generalized convexity and concavity properties in the context of local fractional derivatives. The derived results extend classical inequalities to the framework of fractional calculus, offering new tools for analyzing functions with non-integer order differentiability. Furthermore, by incorporating additional inequalities such as the generalized Hölder inequality, generalized power mean inequality, and its improved version, we have enriched the theoretical foundation of local fractional analysis. The practical applications presented underscore the effectiveness of our findings, contributing valuable tools for real-world mathematical challenges.

Acknowledgement

The authors express their sincere gratitude to the editor and anonymous reviewers for their valuable comments, constructive feedback, and insightful suggestions.

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