J. Inequal. Math. Anal., 1(1), 2025, 65–78

Journal of Inequalities and Mathematical Analysis

ISSN:3062-3251
Journal Homepage: www.inequalmath.com

Research Article

Some Trapezoid-type Inequalities for Newly Defined Proportional Caputo-Hybrid Operator

Tuba Tunç *[a], İzzettin Demir [a]

[a] *Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce 81620, Türkiye.*

Abstract

This study starts with the construction of a novel identity for the proportional Caputo-hybrid operator. Building on this identity, we develop several integral inequalities related to the right-hand side of Hermite-Hadamard-type inequalities in the context of the proportional Caputo-hybrid operator. Additionally, we demonstrate that the proposed results improve and generalize some previously established findings in the domain of integral inequalities. Lastly, in order to clarify and improve comprehension of the recently established inequalities, we provide numerous examples together with their graphical representations.

Keywords: Hermite-Hadamard-type inequalities, Trapezoid-type inequalities, Convex functions, Riemann-Liouville fractional integrals, Proportional Caputo-hybrid operator

2020 MSC: 26D07, 26D10, 26D15, 26A33.

1. Introduction

Convex analysis, as a branch of mathematics, has found extensive applications in diverse fields such as energy systems, physics and engineering applications. In mathematics, this analysis holds a critical importance, especially in the exploration of inequalities. Among the most renowned inequalities in convex theory is the Hermite-Hadamard inequality, initially studied by C. Hermite and J. Hadamard [1], [2] expressed in this manner:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function. For a concave function f, the inequality presented is reversed.

The Hermite-Hadamard inequality offers a way to bound the average value of a convex function over a compact interval. Its relevance extends to diverse areas of mathematics, such as probability, statistics, integral calculus, and optimization theory, and it forms the foundation for various related inequalities. Moreover, it is employed in solving real-world problems in physics, engineering, economics, and other disciplines. With the emergence of new problems, the applications of this inequality keep expanding, establishing it as a crucial instrument for solving extensive mathematical problems as well as problems in other disciplines. On the other hand, with the trapezoid inequality defining its right-hand side and the midpoint inequality its left-hand side, the Hermite-Hadamard inequality has become a central

*Corresponding author. Email: tubatunc03@gmail.com
Email addresses: tubatunc03@gmail.com (Tuba Tunç *), izzettindemir@duzce.edu.tr (İzzettin Demir)

doi: 10.63286/jima.2025.05

Received: 23 January 2025 Revised: 26 February 2025 Accepted: 19 March 2025

topic in inequality studies. The trapezoid-type inequalities for convex functions were pioneered by Dragomir and Agarwal in [3], whereas Kırmaci was the first to address midpoint-type inequalities in [4]. These developments have led to an active area of research in the field of inequalities [5], [6], [7].

As a field concerned with derivatives and integrals of fractional orders, fractional calculus gives a generalized framework that extends classical calculus. It is widely applied in the analysis of complex systems, fractional differential equations, fractal geometry, and diverse scientific and engineering problems [8, 9, 10, 11]. Additionally, this field establishes a comprehensive framework for investigating systems characterized by fractional dynamics, offering a richer mathematical description of complex processes. As a result, fractional calculus has seen growing importance and widespread application in science and engineering, particularly through the development of advanced fractional operators such as Caputo-Fabrizio [12], Atangana-Baleanu [13], and tempered [14].

The Riemann-Liouville integral operators, defined as follows, are regarded as fundamental tools in the study of fractional integrals [15]:

Definition 1.1. The Riemann-Liouville integrals of order $\alpha > 0$ are presented by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

for $f \in L_1[a,b]$. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$. The Riemann-Liouville integrals reduce to classical integrals under the condition $\alpha = 1$.

The Hermite-Hadamard inequality was reinterpreted by Sarıkaya and Yıldırım [16] using fractional integrals, which is presented in the following form:

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $0 \le a < b$ and $f \in L_1[a,b]$. Assuming f is a convex function on $[a,b]$, the following fractional integral inequalities hold true:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J_{\left(\frac{a+b}{2}\right)+}^{\alpha} f(b) + J_{\left(\frac{a+b}{2}\right)-}^{\alpha} f(a)] \le \frac{f(a)+f(b)}{2}$$

with $\alpha > 0$.

Following this, Sarıkaya et al. [17] and Iqbal et al. [18] established various fractional midpoint-type and trapezoid-type inequalities for convex functions, respectively. Further references on this subject can be found in [19, 20].

An additional significant definition in fractional analysis is presented below [21]:

Definition 1.3. Take $\alpha > 0$ and $\alpha \notin \{1,2,...\}$, $n = [\alpha]+1$, $f \in AC^n[a,b]$, the space of functions having n-th derivatives absolutely continuous. Below are the definitions for the Caputo fractional derivatives of order α, both left-sided and right-sided:

$${}^{C}D_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)dt, \quad x > a$$

and

$${}^{C}D_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b (t-x)^{n-\alpha-1} f^{(n)}(t)dt, \quad x < b.$$

If $\alpha = n \in \{1,2,3,...\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo fractional derivative ${}^{C}D_{a^+}^{\alpha} f(x)$ coincides with $f^{(n)}(x)$ whereas ${}^{C}D_{b^-}^{\alpha} f(x)$ with exactness to a constant multiplier $(-1)^n$. If $n = 1$ and $\alpha = 0$, then we get ${}^{C}D_{a^+}^{\alpha} f(x) = {}^{C}D_{b^-}^{\alpha} f(x) = f(x)$.

In the Caputo derivative, a fractional integral is applied to the standard derivative of the function, in contrast to the Riemann-Liouville fractional derivative, which is derived by differentiating the fractional integral of a function for its independent variable of order n. The Caputo fractional derivative requires more appropriate initial conditions compared to the traditional Riemann-Liouville fractional derivative when dealing with fractional differential equations [22]. Therefore, in the context of other fractional derivatives, the Caputo derivative is preferred because it delivers solutions that are more physically interpretable for certain problems. In addition, the operator of proportional derivative symbolized as ${}^{P}D_{\alpha}f(x)$ is given by the equation [23] :

$$ {}^{P}D_{\alpha}f(x) = K_1(\alpha,x)f(x) + K_0(\alpha,x)f'(x), $$

where K_1 and K_0 are the functions satisfying $\alpha \in [0,1]$ and $x \in \mathbb{R}$ with some conditions and also, the function f is differentiable with respect to $x \in \mathbb{R}$. Widely utilized in control systems and robotics, this mathematical operator has been the focus of increasing attention in recent years, particularly in studies involving the Caputo derivative and the proportional derivative [24], [25], [26].

In [27], Baleanu et al. proposed a novel definition by merging the ideas of the Caputo derivative and proportional derivative, creating a hybrid fractional operator represented as a linear combination of the Caputo fractional derivative and the Riemann-Liouville fractional integral.

Definition 1.4. Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I° and f, f' are locally $L_1(I)$. The concept of a proportional Caputo-hybrid operator is given below:

$$ {}_{0}^{PC}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \left[K_1(\alpha,\tau)f(\tau) + K_0(\alpha,\tau)f'(\tau)\right](t-\tau)^{-\alpha} d\tau, $$

where $\alpha \in [0,1]$ and K_1 and K_0 are functions that comply with the conditions given below:

$$ \begin{aligned} \lim_{\alpha\to 0^+} K_0(\alpha,\tau) &= 0; \quad \lim_{\alpha\to 1} K_0(\alpha,\tau) = 1; \quad K_0(\alpha,\tau) \neq 0, \quad \alpha \in (0,1]; \\ \lim_{\alpha\to 0} K_1(\alpha,\tau) &= 0; \quad \lim_{\alpha\to 1^-} K_1(\alpha,\tau) = 1; \quad K_1(\alpha,\tau) \neq 0, \quad \alpha \in [0,1). \end{aligned} $$

Subsequently, Sarıkaya [28] developed an innovative definition by incorporating distinct K_1 and K_0 functions in accordance with Definition 1.4. Also, Sarıkaya [28] established the Hermite-Hadamard inequality using his newly proposed definition, as outlined below:

Definition 1.5. Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I° and $f, f' \in L_1(I)$. The left-sided and right-sided proportional Caputo-hybrid operators of order α are introduced in the following manner, respectively :

$$ {}_{a^+}^{PC}D_b^{\alpha} f(b) = \frac{1}{\Gamma(1-\alpha)} \int_a^b \left[K_1(\alpha,b-\tau)f(\tau) + K_0(\alpha,b-\tau)f'(\tau)\right](b-\tau)^{-\alpha} d\tau $$

and

$$ {}_{b^-}^{PC}D_a^{\alpha} f(a) = \frac{1}{\Gamma(1-\alpha)} \int_a^b \left[K_1(\alpha,\tau-a)f(\tau) + K_0(\alpha,\tau-a)f'(\tau)\right](\tau-a)^{-\alpha} d\tau, $$

where $\alpha \in [0,1]$ and $K_0(\alpha,\tau) = (1-\alpha)^2\tau^{1-\alpha}$ and $K_1(\alpha,\tau) = \alpha^2\tau^{\alpha}$.

Theorem 1.6. *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I°, the interior of the interval I, where $a,b \in I^\circ$ with $a < b$ and let f, f' be the convex functions on I. Then, the following inequalities hold:*

$$ \alpha^2(b-a)^{\alpha} f\left(\frac{a+b}{2}\right) + \frac{1}{2}(1-\alpha)(b-a)^{1-\alpha} f'\left(\frac{a+b}{2}\right) $$

$$\leq \frac{\Gamma(1-\alpha)}{2(b-a)^{1-\alpha}}\left[{}^{PC}_{a^+}D^{\alpha}_{b}\, f(b)+{}^{PC}_{b^-}D^{\alpha}_{a}\, f(a)\right]$$

$$\leq \alpha^2(b-a)^{\alpha}\left[\frac{f(a)+f(b)}{2}\right]+(1-\alpha))(b-a)^{1-\alpha}\left[\frac{f'(a)+f'(b)}{4}\right].$$

On the other hand, Tunç and Demir [29] proposed a Hermite-Hadamard inequality by using the Caputo-hybrid operator in a distinct manner than the previous theorem, as shown below:

Theorem 1.7. *Let $f: I\subset\mathbb{R}^+\to\mathbb{R}$ be a differentiable function on I^o, the interior of the interval I, where $a,b\in I^o$ satisfying $a<b$ and f,f' be the convex functions on I. Then, the following inequalities are satisfied:*

$$\alpha^2(b-a)^{\alpha}2^{-\alpha}f\left(\frac{a+b}{2}\right)+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}f'\left(\frac{a+b}{2}\right) \tag{1.2}$$

$$\leq \frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D^{\alpha}_{b}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D^{\alpha}_{a}f(a)\right]$$

$$\leq \alpha^2(b-a)^{\alpha}2^{-\alpha}\left(\frac{f(a)+f(b)}{2}\right)+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}\left(\frac{f'(a)+f'(b)}{2}\right).$$

In this paper, firstly, we give an identity with respect to the novel proportional Caputo-hybrid operator. This identity serves as a fundamental tool in developing a variety of trapezoid-type inequalities. Then, we obtain several significant inequalities by employing convexity, the Hölder inequality, and the power mean inequality. Furthermore, we offer illustrative examples supported by graphical representations to establish the accuracy of our main results. The results outlined here generalize a number of inequalities reported in previous investigations.

2. Main Results

In order to demonstrate our other primary results, we depend on the following lemma. By taking advantage of this finding, we establish several integral inequalities that are related to the right-hand side of Hermite-Hadamard-type inequalities for proportional Caputo-hybrid operator.

Lemma 2.1. *Let $f: I\subset\mathbb{R}^+\to\mathbb{R}$ be a twice differentiable function on I^o, the interior of the interval I, where $a,b\in I^o$ satisfying $a<b$ and let $f,f',f''\in L[a,b]$. Then, the following identity are satisfied:*

$$\frac{\alpha^2(b-a)^{\alpha+1}2^{-\alpha}}{2}\int_0^1(1-2t)f'(ta+(1-t)b)\,dt \tag{2.1}$$

$$+\frac{(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}}{2}\left\{\int_0^1\left(t^{2-2\alpha}-1\right)\left[f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)-f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt\right\}$$

$$= \alpha^2(b-a)^{\alpha}2^{-\alpha}\left(\frac{f(a)+f(b)}{2}\right)+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}\left(\frac{f'(a)+f'(b)}{2}\right)$$

$$-\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D^{\alpha}_{b}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D^{\alpha}_{a}f(a)\right].$$

Proof. By integration by parts, we have

$$\int_0^1 (t-1)f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt=\frac{2}{b-a}f(a)-\frac{2}{b-a}\int_0^1 f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt$$

and

$$\int_0^1 (t^{2-2\alpha}-1)f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt=\frac{2}{b-a}f'(a)-\frac{4(1-\alpha)}{b-a}\int_0^1 t^{1-2\alpha}f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt.$$

Using a variable substitution, multiplying the results by $\alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}$ and $(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}$, and combining them through addition, we reach the following outcome:

$$\begin{aligned}
&\alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}\int_0^1 (t-1)f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt \\
&+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\int_0^1 (t^{2-2\alpha}-1)f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt \\
=\;&\alpha^2(b-a)^{\alpha}2^{-\alpha}f(a)+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}f'(a) \\
&-\frac{2^{1-\alpha}}{(b-a)^{1-\alpha}}\int_a^{\frac{a+b}{2}}\left[\alpha^2(\tau-a)^{\alpha}f(\tau)+(1-\alpha)^2(\tau-a)^{1-\alpha}f'(\tau)\right](\tau-a)^{-\alpha}d\tau.
\end{aligned}\tag{2.2}$$

By following similar steps, we obtain

$$\begin{aligned}
&\alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}\int_0^1 (t-1)f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt \\
&+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\int_0^1 (t^{2-2\alpha}-1)f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt \\
=\;&-\alpha^2(b-a)^{\alpha}2^{-\alpha}f(b)-(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}f'(b) \\
&+\frac{2^{1-\alpha}}{(b-a)^{1-\alpha}}\int_{\frac{a+b}{2}}^{b}\left[\alpha^2(b-\tau)^{\alpha}f(\tau)+(1-\alpha)^2(b-\tau)^{1-\alpha}f'(\tau)\right](b-\tau)^{-\alpha}d\tau.
\end{aligned}\tag{2.3}$$

Through the extraction of (2.3) from (2.2), we have

$$\alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}\int_0^1 (t-1)\left[f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt$$

$$+\frac{(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}}{2}\int_0^1 (t^{2-2\alpha}-1)\left[f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)-f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt$$

$$= \alpha^2(b-a)^{\alpha}2^{-\alpha}\left(f(a)+f(b)\right)+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}\left(f'(a)+f'(b)\right)$$

$$-\frac{\Gamma(1-\alpha)}{2^{-1+\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D_b^{\alpha}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D_a^{\alpha}f(a)\right].$$

Thus, by multiplying the both sides by $\frac{1}{2}$ and by using the equality

$$\int_0^1 (t-1)\left[f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt=2\int_0^1 (1-2t)f'\left(ta+(1-t)b\right)dt,$$

we derive the conclusion (2.1). □

Remark 2.2. Letting the limit as $\alpha\to 1$ in Lemma 2.1, it follows that

$$\left(\frac{f(a)+f(b)}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx=\frac{(b-a)}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt,$$

which was proved by Dragomir and Agarwal [3].

Corollary 2.3. *In the limiting case $\alpha=0$ in Lemma 2.1, we obtain*

$$\frac{(b-a)^2}{4}\left(\int_0^1 (t^2-1)\left[f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)-f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt\right)$$

$$=\frac{(b-a)}{4}\left(\frac{f'(a)+f'(b)}{2}\right)-\frac{1}{b-a}\left(\int_{\frac{a+b}{2}}^{b} f(x)dx-\int_a^{\frac{a+b}{2}} f(x)dx\right).$$

Moreover, if we choose $\alpha=\frac{1}{2}$, then the equality (2.1) transforms into the following equality

$$\frac{2}{b-a}\left\{\frac{f(a)+f(b)}{2}+\frac{f'(a)+f'(b)}{2}-\left[\int_a^b f(x)dx+f(b)-f(a)\right]\right\}$$

$$=\int_0^1 (1-2t)\left[f'(ta+(1-t)b)+f''(ta+(1-t)b)\right]dt.$$

Theorem 2.4. *Let $f:I\subset\mathbb{R}^+\to\mathbb{R}$ be a twice differentiable function on I^o, the interior of the interval I, where $a,b\in I^o$ satisfying $a<b$ and let $f,f',f''\in L[a,b]$. If $|f'|^q$ and $|f''|^q$ are convex on $[a,b]$ for $q\geq 1$, then the following inequality holds:*

$$\left|\alpha^2(b-a)^{\alpha}2^{-\alpha}\left(\frac{f(a)+f(b)}{2}\right)+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}\left(\frac{f'(a)+f'(b)}{2}\right)\right. \tag{2.4}$$

$$
\begin{aligned}
&-\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^{+}}D^{\alpha}_{b}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^{-}}D^{\alpha}_{a}f(a)\right]\Bigg| \\
\leq\ &\frac{\alpha^{2}(b-a)^{\alpha+1}2^{-\alpha}}{4}\left(\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right)^{\frac{1}{q}} \\
&+\frac{(1-\alpha)(b-a)^{2-\alpha}2^{\alpha}}{16}\left\{\left(\frac{2-2\alpha}{3-2\alpha}\right)^{\frac{q-1}{q}}\left[\left(\frac{|f''(a)|^{q}}{2}\left(\frac{5-6\alpha}{2(3-2\alpha)}+\frac{1}{4-2\alpha}\right)+\frac{|f''(b)|^{q}}{2}\left(\frac{1-\alpha}{4-2\alpha}\right)\right)^{\frac{1}{q}}\right.\right. \\
&\left.\left.+\left(\frac{|f''(a)|^{q}}{2}\left(\frac{1-\alpha}{4-2\alpha}\right)+\frac{|f''(b)|^{q}}{2}\left(\frac{5-6\alpha}{2(3-2\alpha)}+\frac{1}{4-2\alpha}\right)\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}
$$

Proof. Firstly, let $q=1$. By the convexity of $|f'|$ and $|f''|$, from Lemma 2.1 it follows that

$$
\begin{aligned}
&\left|\alpha^{2}(b-a)^{\alpha}2^{-\alpha}\left(\frac{f(a)+f(b)}{2}\right)+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}\left(\frac{f'(a)+f'(b)}{2}\right)\right. \\
&-\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^{+}}D^{\alpha}_{b}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^{-}}D^{\alpha}_{a}f(a)\right]\Bigg| \\
\leq\ &\frac{\alpha^{2}(b-a)^{\alpha+1}2^{-\alpha}}{2}\int_{0}^{1}|1-2t|\left(t|f'(a)|+(1-t)|f'(b)|\right)dt \\
&+\frac{(1-\alpha)(b-a)^{2-\alpha}2^{\alpha}}{16}\int_{0}^{1}\left|t^{2-2\alpha}-1\right|\left(\frac{2-t}{2}\left|f''(a)\right|+\frac{t}{2}\left|f''(b)\right|+\frac{t}{2}\left|f''(a)\right|+\frac{2-t}{2}\left|f''(b)\right|\right)dt.
\end{aligned}
\tag{2.5}
$$

Hence, due to

$$
\int_{0}^{\frac{1}{2}}(1-2t)tdt=\int_{\frac{1}{2}}^{1}(2t-1)(1-t)dt=\frac{1}{24},\qquad \int_{0}^{\frac{1}{2}}(1-2t)(1-t)dt=\int_{\frac{1}{2}}^{1}(2t-1)tdt=\frac{5}{24}
$$

and

$$
\int_{0}^{1}\left(1-t^{2-2\alpha}\right)dt=\frac{2-2\alpha}{3-2\alpha},
$$

we arrive at the expression on the right-hand side of inequality (2.5) in the following manner:

$$
\frac{\alpha^{2}(b-a)^{\alpha+1}2^{-\alpha}}{4}\left(\frac{f'(a)+f'(b)}{2}\right)+\frac{(1-\alpha)(b-a)^{2-\alpha}2^{\alpha}}{4}\left(\frac{1-\alpha}{3-2\alpha}\right)\left(\frac{f''(a)+f''(b)}{2}\right).
$$

Moreover, for $q>1$, based on Lemma 2.1, utilizing the power mean inequality and the convexity of $|f'|^{q}$ and $|f''|^{q}$, we can derive:

$$
\left|\alpha^{2}(b-a)^{\alpha}2^{-\alpha}\left(\frac{f(a)+f(b)}{2}\right)+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}\left(\frac{f'(a)+f'(b)}{2}\right)\right.
$$

$$- \frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}} \left[{}^{PC}_{\left(\frac{a+b}{2}\right)^{+}}D_b^{\alpha} f(b) + {}^{PC}_{\left(\frac{a+b}{2}\right)^{-}}D_a^{\alpha} f(a) \right] \Bigg|$$

$$\leq \frac{\alpha^2 (b-a)^{\alpha+1} 2^{-\alpha}}{2} \left\{ \left(\int_0^1 |1-2t|\, dt \right)^{\frac{1}{p}} \left(\int_0^1 |1-2t| \left[t|f'(a)|^q + (1-t)|f'(b)|^q \right] dt \right)^{\frac{1}{q}} \right\}$$

$$+ \frac{(1-\alpha)(b-a)^{2-\alpha} 2^{\alpha}}{16} \left\{ \left(\int_0^1 \left(1-t^{2-2\alpha}\right) dt \right)^{\frac{1}{p}} \left(\int_0^1 \left(1-t^{2-2\alpha}\right) \left[\frac{t}{2}|f''(a)|^q + \frac{2-t}{2}|f''(b)|^q \right] dt \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_0^1 1-t^{2-2\alpha} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left(1-t^{2-2\alpha}\right) \left[\frac{2-t}{2}|f''(a)|^q + \frac{t}{2}|f''(b)|^q \right] dt \right)^{\frac{1}{q}} \right\}.$$

Thus, because of

$$\int_0^1 |1-2t|\, dt = \int_0^{\frac{1}{2}} (1-2t)\, dt + \int_{\frac{1}{2}}^1 (2t-1)\, dt = \frac{1}{2}, \qquad \int_0^{\frac{1}{2}} (1-2t) t dt = \int_{\frac{1}{2}}^1 (2t-1)(1-t) dt = \frac{1}{24},$$

$$\int_0^1 \left(1-t^{2-2\alpha}\right) dt = \frac{2-2\alpha}{3-2\alpha}, \qquad \int_0^{\frac{1}{2}} (1-2t)(1-t) dt = \int_{\frac{1}{2}}^1 (2t-1) t dt = \frac{5}{24}$$

and

$$\int_0^1 \left(1-t^{2-2\alpha}\right)(2-t)\, dt = \frac{5-6\alpha}{6-4\alpha} + \frac{1}{4-2\alpha}, \qquad \int_0^1 \left(1-t^{2-2\alpha}\right) t dt = \frac{1-\alpha}{4-2\alpha},$$

the desired inequality (2.4) is satisfied. □

Remark 2.5. Letting the limit as $\alpha \to 1$ and putting $q = 1$ in Theorem 2.4, it follows that

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)}{4} \left(\frac{|f'(a)| + |f'(b)|}{2} \right),$$

which was proved by Dragomir and Agarwal [3].

Corollary 2.6. *As α approaches* 0 *and for $q \geq 1$ in Theorem 2.4, we obtain*

$$\left| \frac{(b-a)}{4} \left(\frac{f'(a)+f'(b)}{2} \right) - \frac{1}{b-a} \left(\int_{\frac{a+b}{2}}^b f(x) dx - \int_a^{\frac{a+b}{2}} f(x) dx \right) \right|$$

$$\leq \frac{(b-a)^2}{16} \left(\frac{2}{3} \right)^{\frac{q-1}{q}} \left[\left(\frac{13}{24} \left| f''(a) \right|^q + \frac{1}{8} \left| f''(b) \right|^q \right)^{1/q} + \left(\frac{1}{8} \left| f''(a) \right|^q + \frac{13}{24} \left| f''(b) \right|^q \right)^{1/q} \right].$$

Furthermore, when α converges to 1 *and under the condition $q \geq 1$, the inequality in Theorem 2.4 takes the following form:*

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ \leq \; & \frac{(b-a)^2}{8} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q}. \end{aligned}$$

To validate the effectiveness of our theorem, we present an illustrative example.

Example 2.7. Taking the function $f(x) = x^3 + x$ defined on the interval $[0,2]$, we can evaluate the right-hand side of inequality (2.4) as:

$$\frac{\alpha^2}{2}\left(\frac{1+13^q}{2}\right)^{\frac{1}{q}} + \frac{(1-\alpha)}{4}\left(\frac{2-2\alpha}{3-2\alpha}\right)^{\frac{q-1}{q}} \frac{12}{2^{\frac{1}{q}}}\left[\left(\frac{1-\alpha}{4-2\alpha}\right)^{\frac{1}{q}} + \left(\frac{5-6\alpha}{6-4\alpha} + \frac{1}{4-2\alpha}\right)^{\frac{1}{q}}\right].$$

On the other hand, we observe that

$$\begin{aligned} & \left| \alpha^2 (b-a)^{\alpha} 2^{-\alpha} \left(\frac{f(a)+f(b)}{2} \right) + (1-\alpha)(b-a)^{1-\alpha} 2^{\alpha-2} \left(\frac{f'(a)+f'(b)}{2} \right) \right. \\ & \left. - \frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}} \left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+} D_b^{\alpha} f(b) + {}^{PC}_{\left(\frac{a+b}{2}\right)^-} D_a^{\alpha} f(a) \right] \right| \\ = \; & 2\alpha^2 + \frac{7}{2}(1-\alpha) - \frac{(1-\alpha)^2}{2} \left(\frac{14}{2-2\alpha} - \frac{12}{3-2\alpha} + \frac{6}{4-2\alpha} \right). \end{aligned}$$

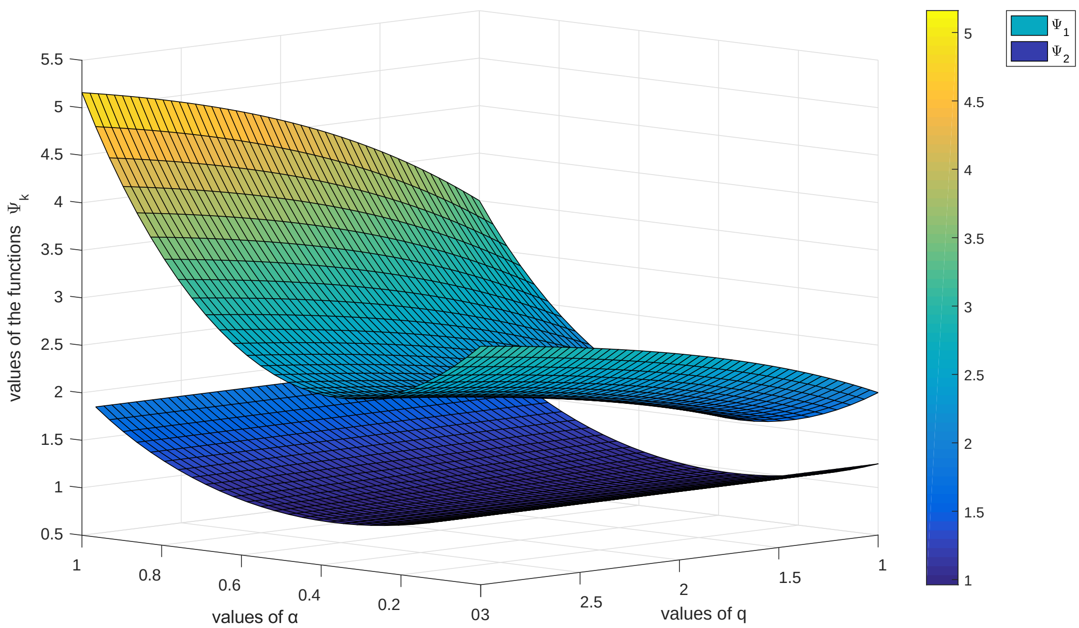

Figure 1: The graph of both sides of the inequality (2.4) depending on $\alpha \in (0,1)$ and $q \in [1,3]$.

Figure 1 clearly demonstrates that the left-hand side of the inequality (2.4) is consistently situated below the right-hand side of this inequality for all $\alpha \in (0,1)$ and $q \geq 1$.

Theorem 2.8. *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function on I^o, the interior of the interval I, where $a, b \in I^o$ satisfying $a < b$ and let $f, f', f'' \in L[a,b]$. If $|f'|^q$ and $|f''|^q$ are convex on $[a,b]$ for $q > 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\alpha^2(b-a)^{\alpha}2^{-\alpha}\left(\frac{f(a)+f(b)}{2}\right)+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}\left(\frac{f'(a)+f'(b)}{2}\right)\right. \\ &\left.-\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D^{\alpha}_{b}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D^{\alpha}_{a}f(a)\right]\right| \\ \leq\ &\frac{\alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}}{(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}} \\ &+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-4}\left(\frac{(2-2\alpha)p}{(2-2\alpha)p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f''(a)|^q+3|f''(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f''(a)|^q+|f''(b)|^q}{4}\right)^{\frac{1}{q}}\right], \end{aligned} \tag{2.6}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Applying the commonly used Hölder's inequality and the convexity of $|f'|^q$, $|f''|^q$, from Lemma 2.1, we obtain

$$\begin{aligned} &\left|\alpha^2(b-a)^{\alpha}2^{-\alpha}\left(\frac{f(a)+f(b)}{2}\right)+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}\left(\frac{f'(a)+f'(b)}{2}\right)\right. \\ &\left.-\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D^{\alpha}_{b}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D^{\alpha}_{a}f(a)\right]\right| \\ \leq\ &\alpha^2(b-a)^{\alpha+1}2^{-\alpha-2}\left[\left(\int_0^1|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right] \\ &+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-4}\left[\left(\int_0^1\left|t^{2-2\alpha}-1\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left[\frac{t}{2}\left|f''(a)\right|^q+\frac{2-t}{2}\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\int_0^1\left|t^{2-2\alpha}-1\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left[\frac{2-t}{2}\left|f''(a)\right|^q+\frac{t}{2}\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{2.7}$$

Calculating the integrals in the above inequality, we have

$$\int_0^1|1-2t|^p\,dt = \int_0^{1/2}(1-2t)^p dt+\int_{1/2}^1(2t-1)^p dt=\frac{1}{p+1},$$

$$\int_0^1\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt = \frac{|f'(a)|^q+|f'(b)|^q}{2},$$

$$\int_0^1 \left[\frac{t}{2}\left|f''(a)\right|^q+\frac{2-t}{2}\left|f''(b)\right|^q\right]dt \quad = \quad \frac{|f''(a)|^q+3|f''(b)|^q}{4},$$

$$\int_0^1 \left[\frac{2-t}{2}\left|f''(a)\right|^q+\frac{t}{2}\left|f''(b)\right|^q\right]dt \quad = \quad \frac{3|f''(a)|^q+|f''(b)|^q}{4}.$$

Also, using the property that is $(A-B)^p \leq A^p - B^p$ for $A > B \geq 0$ and $p \geq 1$, we get

$$\int_0^1 \left|t^{2-2\alpha}-1\right|^p dt \leq \int_0^1 \left[1-t^{(2-2\alpha)p}\right] dt = \frac{(2-2\alpha)p}{[(2-2\alpha)p+1]}.$$

Replacing the derived integral results in the inequality (2.7), the desired result will be achieved. □

Remark 2.9. In the special case where α approaches 1 in Theorem 2.8, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2\,(p+1)^{1/p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q},$$

which was presented by Dragomir and Agarwal in [3].

Corollary 2.10. *When α approaches 0 in Theorem 2.8, we have*

$$\left|\frac{(b-a)}{4}\left(\frac{f'(a)+f'(b)}{2}\right)-\frac{1}{b-a}\left(\int_{\frac{a+b}{2}}^{b} f(x)dx-\int_a^{\frac{a+b}{2}} f(x)dx\right)\right|$$

$$\leq \quad \frac{(b-a)^2}{16}\left(\frac{2p}{2p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f''(a)|^q+3|f''(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f''(a)|^q+|f''(b)|^q}{4}\right)^{\frac{1}{q}}\right].$$

Also, if we take $\alpha = \frac{1}{2}$, then we obtain

$$\left|\frac{f(a)+f(b)}{2}+\frac{f'(a)+f'(b)}{2}-\frac{1}{b-a}\left[\int_a^b f(x)dx+f(b)-f(a)\right]\right|$$

$$\leq \quad \frac{b-a}{2(p+1)^{1/p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}$$

$$+\frac{b-a}{4}\left(\frac{p}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f''(a)|^q+3|f''(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f''(a)|^q+|f''(b)|^q}{4}\right)^{\frac{1}{q}}\right].$$

In order to demonstrate the inequality derived in Theorem 2.8, we present an example that verifies its validity.

Example 2.11. Let us consider the function f as defined in Example 2.7. Later, we can evaluate the expression on the right-hand side of the inequality (2.6) in the following way:

$$\frac{\alpha^2}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{1+13^{\frac{p}{p-1}}}{2}\right)^{\frac{p-1}{p}}\right]+\frac{3(1-\alpha)}{4^{\frac{p-1}{p}}}\left(\frac{(2-2\alpha)p}{[(2-2\alpha)p+1]}\right)^{\frac{1}{p}}\left(1+3^{\frac{p-1}{p}}\right).$$

Moreover, we know that

$$
\begin{aligned}
&\left|\alpha^2(b-a)^{\alpha}2^{-\alpha}\left(\frac{f(a)+f(b)}{2}\right)+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}\left(\frac{f'(a)+f'(b)}{2}\right)\right.\\
&\left.-\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D_b^{\alpha}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D_a^{\alpha}f(a)\right]\right|\\
=\;& 2\alpha^2+\frac{7}{2}(1-\alpha)-\frac{(1-\alpha)^2}{2}\left(\frac{14}{2-2\alpha}-\frac{12}{3-2\alpha}+\frac{6}{4-2\alpha}\right).
\end{aligned}
$$

Therefore, it can be seen from Figure 2 that the left-hand side of inequality (2.6) is consistently lower than the right-hand side for all values of $\alpha \in (0,1)$ and $p > 1$.

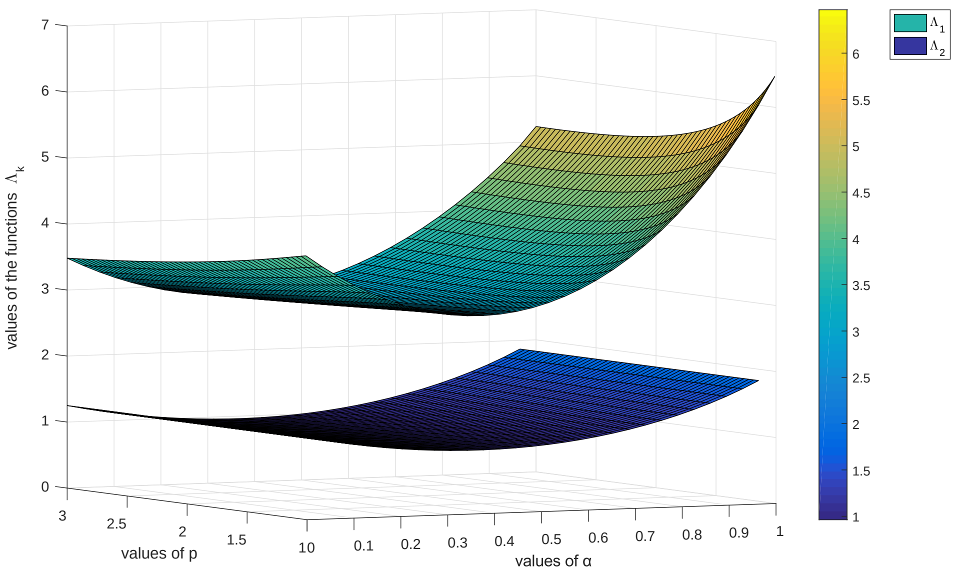

Figure 2: The graph of both sides of the inequality (2.6) depending on $\alpha \in (0,1)$ and $p \in (1,3]$.

3. Conclusion

This paper begins with the development of a novel identity for the proportional Caputo-hybrid operator. Based on this identity, we establish several integral inequalities associated with the right-hand side of Hermite-Hadamard-type inequalities in the framework of the proportional Caputo-hybrid operator. Furthermore, we show that the proposed results refine and generalize certain previously established results in the field of integral inequalities. Finally, to enhance understanding and clarity of the newly derived inequalities, we provide several examples along with their graphical illustrations. In comparison to classical calculus, our results are more advantageous as they highlight the specific case of previously established bounds when α approaches 1. We hope that our approach and results will inspire readers to explore this topic further. Future studies may explore similar inequalities for different fractional integrals, and by considering other types of convexity, new Hermite-Hadamard-type inequalities could be derived.

Funding

There is no funding.

Availability of Data Materials

Data sharing not applicable to this paper as no data sets were generated or analysed during the current study.

Conflict interests

The authors declare that they have no conflict interests.

References

[1] S.S. Dragomir and C.E.M. Pearce, *Selected topics on the Hermite-Hadamard inequalities and applications*, RGMIA Monographs, Victoria University.

[2] J. Hadamard, *Etude sur les proprietes des fonctions entieres et en particulier d'une fonction considree par*, Riemann, Journal de Mathématiques Pures et Appliquées, 58, 1893, 171-215.

[3] S.S. Dragomir and R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Applied Mathematics Letters, 11(5), 1998, 91-95. DOI: 10.1016/S0893-9659(98)00086-X

[4] U.S. Kırmacı, *Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula*, Applied Mathematics and Computation, 147(5), 2004, 137-146. DOI: 10.1016/S0096-3003(02)00657-4

[5] N. Alp, M.Z. Sarıkaya, M. Kunt, and İ. İşcan, *q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions,* Journal of King Saud University-Science, 30(2), 2018, 193-203. DOI: 10.1016/j.jksus.2016.09.007

[6] H. Budak, T. Tunç, and M.Z. Sarıkaya, *Fractional Hermite-Hadamard-type inequalities for interval-valued functions,* Proceedings of the American Mathematical Society, 148 (2), 2020, 705-718. DOI: 10.1090/proc/14741

[7] F. Hezenci, H. Kara, and H. Budak, *Conformable fractional versions of Hermite-Hadamard-type inequalities for twice-differentiable functions,* Boundary Value Problems, 2023(48), 2023, 16 pages. DOI: 10.1186/s13661-023-01737-y

[8] M.Z. Javed, M.U. Awan, L. Ciurdariu, S.S. Dragomir, and Y. Almalki, *On extended class of totally ordered interval-valued convex stochastic processes and applications*, Fractal and Fractional, 8(10), 2024, 577. DOI: 10.3390/fractalfract8100577

[9] M.Z. Javed, M.U. Awan, B. Bin-Mohsin, and S. Treanta, *Upper bounds for the remainder term in Boole's quadrature rule and applications to numerical analysis*, Mathematics, 12(18), 2024, 2920. DOI: 10.3390/math12182920

[10] M.Z. Javed, M.U. Awan, B. Bin-Mohsin, H. Budak, and S.S. Dragomir, *Some classical inequalities associated with generic identity and applications*, Axioms, 13(8), 2024, 533. DOI: 10.3390/axioms13080533

[11] B. Bin-Mohsin, M.Z. Javed, M.U. Awan and A. Kashuri, *On some new AB-fractional inclusion relations*, Axioms, 7(10), 2023, 725. DOI: 10.3390/fractalfract7100725

[12] M. Caputo and M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Progress in Fractional Differentiation and Applications, 1(2), 2015, 73-85. DOI: 10.12785/pfda/010201

[13] A. Atangana and D. Baleanu, *New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model*, Thermal Science, 20(2), 2016, 763-769. DOI: 10.48550/arXiv.1602.03408

[14] F. Sabzikar, M.M. Meerschaert, and J. Chen, *Tempered fractional calculus,* Journal of Computational Physics, 293, 2015, 14-28. DOI: 10.1016/j.jcp.2014.04.024

[15] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.

[16] M.Z. Sarıkaya and H. Yıldırım, *On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals*, Miskolc Mathematical Notes, 17(2), 2017, 1049-1059. DOI: 10.18514/MMN.2017.1197

[17] M.Z. Sarıkaya, E. Set, H. Yaldız, and N. Basak, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, 57(9-10), 2013, 2403-2407. DOI: 10.1016/j.mcm.2011.12.048

[18] M. Iqbal, B.M. Iqbal, and K. Nazeer, *Generalization of Inequalities Analogous to Hermite-Hadamard Inequality via Fractional Integrals,* Bulletin of the Korean Mathematical Society, 52(3), 2015, 707-716. DOI: 10.4134/BKMS.2015.52.3.707

[19] H. Budak, F. Ertugral, and M.Z. Sarıkaya, *New generalization of Hermite-Hadamard type inequalities via generalized fractional integrals,* Analele Universitatii din Craiova. Seria Matematica-Informatica, 47(2), 2020, 369-386. DOI: 10.52846/ami.v47i2.1309

[20] P.O. Mohammed and I. Brevik, *A new version of the Hermite–Hadamard inequality for Riemann–Liouville fractional integrals*, Symmetry, 12(4), 2020, 11 pages. DOI: 10.3390/sym12040610

[21] S. Samko, A. Kilbas, and O. Marichev, *Fractional integrals and derivatives: theory and applications*, Gordon and Breach, London, 1993.

[22] K. Diethelm, *The analysis of fractional differential equations: an application-oriented exposition using differential operators of Caputo type*, Springer: Berlin, Germany, 2010.

[23] D.R. Anderson and D.J. Ulness, *Newly defined conformable derivatives*, Advances in Dynamical Systems and Applications, 10(2), 2015, 109-137.

[24] H. Günerhan, H. Dutta, M.A. Dokuyucu, and W. Adel, *Analysis of a fractional HIV model with Caputo and constant proportional Caputo operators,* Chaos, Solitons and Fractals, 139, 2020, 19 pages. DOI: 10.1016/j.chaos.2020.110053

[25] R. Hajaj and Z. Odibat, *Numerical solutions of fractional epidemic models with generalized Caputo-type derivatives,* Physica Scripta, 98(4), 2023. DOI: 10.1088/1402-4896/acbfef

[26] G. Rahman, K.S. Nisar, and T. Abdeljawad, *Certain Hadamard proportional fractional integral inequalities*, Mathematics, 8(4), 2020. DOI: 10.3390/math8040504

[27] D. Balenau, A. Ferrandez, and A. Akgül, *On a fractional operator combining proportional and classical differintegrals*, Mathematics, 8(3), 2020, 13 pages. DOI: 10.3390/math8030360

[28] M.Z. Sarıkaya, *On Hermite-Hadamard type inequalities for proportional Caputo-Hybrid Operator*, Konuralp Journal of Mathematics, 11(1), 2023, 31-39.

[29] T. Tunç and İ. Demir, *On a new version of Hermite-Hadamard-type inequality based on proportional Caputo-Hybrid operator*, Boundary Value Problems, 2024(44), 2024. DOI: 10.1186/s13661-024-01852-4