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Research Article

On Some Bullen-type Inequalities with the kth Power for Twice Differentiable Mappings and Applications

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Abstract

We present the generalization of power-mean inequality. Further, we establish some Bullen-type inequalities with the kth power for twice differentiable mappings via s-convex functions in the first and second sense and s-concave functions, using the generalized Hölder's inequality and the generalized power-mean inequality. Some applications to special means of real numbers and to polygamma functions $\psi^{(m)}(x)$for $m = 0, 1, 2, 3$ are also given.

Keywords: Generalized Hölder's inequality, Generalized power-mean inequality, s-convex functions, s-concave functions, Special means.

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1. Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

is known as Hermite-Hadamard's inequality for convex functions. Both inequalities hold in the reversed direction if f is concave [1].
In [2] and [1, p. 278], the following concept was introduced by Orlicz.
$f : [0,\infty) \to R$ is called s-convex in the first sense if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda^s) f(y),$$

holds for all $x, y \in [0,\infty)$, $\lambda \in [0,1]$, and for some fixed $s \in (0,1]$. The class of s-convex functions in the first sense is usually denoted with K_s^1.
In [3] and [1, p. 288], Hudzik and Maligranda considered, among others, the class of functions which are s-convex in the second sense. This class is defined in the following way:
A function $f : [0,\infty) \to \mathbb{R}$ is called s-convex in the second sense if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y),$$

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holds for all $x,y \in [0,\infty)$, $\lambda \in [0,1]$, and for some fixed $s \in (0,1]$. The class of s-convex functions in the second sense is usually denoted by K_s^2.
In [4], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense:

Theorem 1.1. *Suppose that $f : [0,\infty) \to [0,\infty)$ is a s-convex function in the second sense, where $s \in (0,1)$, and let $a,b \in [0,\infty)$ with $a < b$. If $f \in L_1[a,b]$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1}, \tag{1.2}$$

where the constant $k = 1/(s+1)$ is the best possible in the second inequality in (1.2).

In [5], Tseng et al. gave the following result, which is a refinement of (1.1):

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &\le \frac{1}{b-a}\int_a^b f(x)dx \\ &\le \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right] \\ &\le \frac{f(a)+f(b)}{2}. \end{aligned}$$

In [6], Bullen proved the following inequality, which is known as Bullen's inequality, for the convex function f,

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right].$$

In [7], the author gave the following generalization of Hölder's inequality:

Theorem 1.2. *If $a_k,\ b_k \ge 0$ for $k = 1,2,\ldots,n$ and $\frac{1}{sp}+\frac{1}{sq} = 1$ with $p > 1,\ \ s \ge 1$, then*

$$\sum_{k=1}^n (a_k b_k)^{1/s} \le \left(\sum_{k=1}^n a_k^p\right)^{1/sp}\left(\sum_{k=1}^n b_k^q\right)^{1/sq},$$

with equality holding if and only if $\alpha a_k^p = \beta b_k^q$ for $k = 1,2,\ldots,n$, where α and β are real nonnegative constants such that $\alpha^2+\beta^2 > 0$.

In [7, Remark 2.2, e)], the author gave another generalization of Hölder's inequality, for $p > 1$, $s \ge 1$, and $\frac{1}{p}+\frac{1}{q} = 1$:

$$\sum_{k=1}^n (a_k b_k)^s \le \left(\sum_{k=1}^n a_k b_k\right)^s \le \left(\sum_{k=1}^n a_k^p\right)^{s/p}\left(\sum_{k=1}^n b_k^q\right)^{s/q}. \tag{1.3}$$

In [8] and [9], the author gave some inequalities for twice differentiable mappings, using Chebyshev's integral inequality and Grüss inequality. In [10], the author presented some Hermite-Hadamard-type inequalities for twice differentiable m-convex functions and (α,m)-convex functions and gave some applications to special means of real numbers. In [11], Z. Liu gave some Ostrowski-type inequalities for s-convex functions in the second sense. In [12], J. Park obtained new estimates on generalizations of Hermite-Hadamard-like type inequalities for functions whose second derivatives in absolute value at certain powers are convex and concave. In [13], Y. M. Liao et al. established some interesting Riemann-Liouville fractional Hermite-Hadamard inequalities for twice differentiable geometric-arithmetically s-convex functions via the beta function and the incomplete beta function. In [14], B.Y. Xi and F. Qi offered some new inequalities for differentiable convex functions, which are connected with the Hermite-Hadamard integral inequality and applied these inequalities to special means of two positive numbers. In [15], A. Fahad et al. established a new auxiliary identity of the Bullen type for twice-differentiable functions in terms of fractional integral operators and obtained, based on this new identity, some generalized Bullen-type inequalities by employing convexity properties. They gave some concrete examples to illustrate the results and confirmed the correctness by graphical analysis. According to calculations, they showed that improved Hölder and power-mean inequalities give better upper-bound results than classical inequalities. Lastly, they provided some applications to quadrature rules, modified Bessel functions, and digamma functions. In [16], S. Hussain and S. Mehboob established a generalized fractional integral identity to deduce new estimates for the Bullen-type functional and some related inequalities to provide applications in probability and information theory for (s,p)-convex functions by using basic techniques of analysis. In [17], İ. İşcan et al. gave a new general identity for differentiable functions. As a consequence of the identity, they obtained some new general inequalities containing all of the Hermite-Hadamard and Bullen-type inequalities for functions whose derivatives in absolute value at certain powers are convex. They also provided some applications to special means of real numbers. Finally, they addressed some error estimates for the trapezoidal and midpoint formulas.

In this paper, we present some Bullen-type inequalities involving the kth power for twice differentiable mappings via s-convex functions in the first and second sense, as well as s-concave functions, using the generalized Hölder's inequality and the generalized power-mean inequality. Furthermore, we provide some applications to special means of real numbers and to polygamma functions $\psi^{(m)}(x)$ for $m=0,1,2,3$.

Convexity has played a fundamental role in the development of both pure and applied mathematics.

Integral inequalities have significant applications in various fields, including linear programming, combinatorics, orthogonal polynomials, quantum theory, number theory, optimization theory, dynamical systems, and the theory of relativity.

For several recent results concerning Hermite-Hadamard-type inequalities, we refer the reader to [1]-[46].

2. Main Results

First, we give the integral analogue of the generalized Hölder's inequality (1.3).

Theorem 2.1. *Let $p>1$, $s\geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on the interval $[a,b]$ and if $|f|^p$ and $|g|^q$ are integrable functions on $[a,b]$, then*

$$\int_a^b |f(x)g(x)|^s dx \leq \left(\int_a^b |f(x)g(x)|\,dx\right)^s \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{s}{p}} \left(\int_a^b |g(x)|^q dx\right)^{\frac{s}{q}}, \tag{2.1}$$

with equality holding if and only if $A|f(x)|^p = B|g(x)|^q$ almost everywhere, where A and B are constants.

Now, we write a generalization of the power-mean inequality.

Theorem 2.2. *Let $q\geq 1$. If f and g are real functions defined on the interval $[a,b]$ and if $|f|$ and $|f|\,|g|^q$ are integrable functions on $[a,b]$, then for $s\geq 1$, we have*

$$\int_a^b |f(x)g(x)|^s dx \leq \left(\int_a^b |f(x)|\,dx\right)^{s-\frac{s}{q}} \left(\int_a^b |f(x)|\,|g(x)|^q dx\right)^{\frac{s}{q}}. \tag{2.2}$$

Proof. From $\frac{1}{p}+\frac{1}{q}=1$, we have $\frac{s}{p}=s-\frac{s}{q}$, for $s\geq 1$. Using this equality and taking $|f|\,|g| = \left(|f|^{1/p}\right)\left(|f|^{1/q}|g|\right)$ in the generalized Hölder's inequality (2.1), we obtain the generalized power-mean inequality (2.2).

Further, we give the following lemma.

Lemma 2.3. *Let $f : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I^0, where $a,b \in I^0$ with $a < b$. If $f'' \in L_1[a,b]$, then we have*

$$\frac{(b-a)^2}{2}(I_1+I_2) = \frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right],$$

where

$$I_1 = \int_0^{1/2} t\left(t-\frac{1}{2}\right) f''(ta+(1-t)b)\,dt,$$

$$I_2 = \int_{1/2}^{1}\left(t-\frac{1}{2}\right)(t-1) f''(ta+(1-t)b)\,dt.$$

Here, I^0 denotes the interior of I.

Proof. Integrating by parts twice and using the change of the variable $x = ta+(1-t)b$, we have

$$\begin{aligned} I_1 &= \int_0^{1/2} t\left(t-\frac{1}{2}\right) f''(ta+(1-t)b)\,dt \\ &= \frac{-1}{2(a-b)^2}\left[f\left(\frac{a+b}{2}\right) + f(b)\right] + \frac{2}{(a-b)^3}\int_b^{\frac{a+b}{2}} f(s)ds. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} I_2 &= \int_{1/2}^{1}\left(t-\frac{1}{2}\right)(t-1) f''(ta+(1-t)b)\,dt \\ &= \frac{-1}{2(a-b)^2}\left[f(a) + f\left(\frac{a+b}{2}\right)\right] + \frac{2}{(a-b)^3}\int_{\frac{a+b}{2}}^{a} f(s)ds. \end{aligned}$$

Hence, we get

$$I_1+I_2 = \frac{-1}{2(a-b)^2}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right] + \frac{2}{(b-a)^3}\int_a^b f(s)ds.$$

Thus,

$$\frac{(b-a)^2}{2}(I_1+I_2) = \frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right],$$

which completes the proof. In the following theorems, we write some Bullen-type inequalities with the kth power for twice differentiable mappings via s-convex mappings in the first and second sense and s-concave mappings, using the generalized Hölder's inequality and the generalized power-mean inequality.

Theorem 2.4. *Let $f : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I^0 such that $f'' \in L_1[a,b]$, where $q > k \geq 1$ and $a,b \in I^o$ with $a < b$. If the mapping $|f''|^q$ is convex on $[a,b]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right|^k \leq \frac{(b-a)^{2k}}{2}\left(\frac{3}{8}\right)^{k/q}(\alpha^{k/p}+\beta^{k/p})(|f''(a)|^k + |f''(b)|^k).$$

where

$$\alpha \leq \frac{1}{2^{p+2}}\left(\frac{1}{2p+1}+\frac{1}{p+1}\right),$$

$$\beta \leq \frac{2^{2p+1}-1}{8(2p+1)} + \frac{2^{2p+1}+2^{p+1}+p2^p-1}{8(p+1)}.$$

Proof. Using Lemma 2.3, by the generalized Hölder's inequality (2.1) and by the inequality

$$|a+b|^k \leq 2^{k-1}\left(|a|^k+|b|^k\right),$$

for $q>k\geq 1,\ a,b\in\mathbb{R}$, we get

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^k \\
&\leq \frac{(b-a)^{2k}}{2^k}\left[\int_0^{1/2}|t|\left|t-\frac{1}{2}\right|\left|f''(ta+(1-t)b)\right|dt\right. \\
&\quad\left.+\int_{1/2}^1\left|t-\frac{1}{2}\right||t-1|\left|f''(ta+(1-t)b)\right|dt\right]^k \qquad (2.3)\\
&\leq \frac{(b-a)^{2k}}{2}\left[\left(\int_0^{1/2}t^p\left|t-\frac{1}{2}\right|^p dt\right)^{\frac{k}{p}}\left(\int_0^{1/2}\left|f''(ta+(1-t)b)\right|^q dt\right)^{\frac{k}{q}}\right. \\
&\quad\left.+\left(\int_{1/2}^1\left|t-\frac{1}{2}\right|^p|t-1|^p dt\right)^{\frac{k}{p}}\left(\int_{1/2}^1\left|f''(ta+(1-t)b)\right|^q dt\right)^{\frac{k}{q}}\right], \qquad (2.4)
\end{aligned}$$

where $1/p+1/q=1$. Using the convexity of $|f''|^q$, we obtain,

$$\int_0^{1/2}\left|f''(ta+(1-t)b)\right|^q dt\leq\frac{|f''(a)|^q+3|f''(b)|^q}{8},$$

and

$$\int_{1/2}^1\left|f''(ta+(1-t)b)\right|^q dt\leq\frac{3|f''(a)|^q+|f''(b)|^q}{8}.$$

Again, using the fact that

$$|a+b|^p\leq 2^{p-1}\left(|a|^p+|b|^p\right),$$

for $p\geq 0$, and $a,b\in\mathbb{R}$, we obtain

$$\alpha=\int_0^{1/2}t^p\left|t-\frac{1}{2}\right|^p dt\leq\frac{1}{2^{p+2}}\left(\frac{1}{2p+1}+\frac{1}{p+1}\right). \qquad (2.5)$$

$$\begin{aligned}
\beta &= \int_{1/2}^1\left|t-\frac{1}{2}\right|^p|t-1|^p dt \\
&\leq 2^{2p-2}\int_{1/2}^1\left(t^p+\frac{1}{2^p}\right)\left(t^p+1\right)dt \\
&\leq 2^{2p-2}\left[\int_{1/2}^1 t^p\left(t^p+\frac{1}{2^p}\right)dt+\int_{1/2}^1\left(t^p+\frac{1}{2^p}\right)dt\right] \\
&\leq \frac{2^{2p+1}-1}{8(2p+1)}+\frac{2^{2p+1}+2^{p+1}+p2^p-1}{8(p+1)}. \qquad (2.6)
\end{aligned}$$

From the inequalities (2.4)-(2.6), we get

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^k$$
$$\leq \frac{(b-a)^{2k}}{2}\left[\alpha^{k/p}\left(\frac{|f''(a)|^q+3|f''(b)|^q}{8}\right)^{k/q}+\beta^{k/p}\left(\frac{3|f''(a)|^q+|f''(b)|^q}{8}\right)^{k/q}\right].$$

Here $0<\frac{k}{q}<1$, for $q>\ k\geq 1$. Using the fact that

$$\sum_{k=1}^{n}(a_k+b_k)^s\leq\sum_{k=1}^{n}a_k^s+\sum_{k=1}^{n}b_k^s,\quad \text{for } (0\leq s<1),$$
$$a_1,a_2,\dots,a_n\geq 0,\quad b_1,b_2,\dots,b_n\geq 0, \tag{2.7}$$

we obtain

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^k\leq\frac{(b-a)^{2k}}{2}\left(\frac{3}{8}\right)^{k/q}(\alpha^{k/p}+\beta^{k/p})$$
$$\times(|f''(a)|^k+|f''(b)|^k),$$

which is required.

Theorem 2.5. *Let $f:I^0\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I^0 such that $f''\in L_1[a,b]$, where $p>k\geq 1$ and $a,b\in I^o$ with $a<b$. If the mapping $|f''|^p$ is convex on $[a,b]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^k\leq\frac{(b-a)^{2k}}{2\cdot 48^k}\left(\frac{3^{k/p}+1}{4^{k/p}}\right)\left[|f''(a)|^k+|f''(b)|^k\right].$$

Proof. From the inequality (2.3), we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^k\leq\frac{(b-a)^{2k}}{2}\left[\left(\int_0^{1/2}t\left|t-\frac{1}{2}\right||f''(ta+(1-t)b)|dt\right)^k\right.$$
$$\left.+\left(\int_{1/2}^1\left|t-\frac{1}{2}\right||t-1||f''(ta+(1-t)b)|dt\right)^k\right].$$

By the generalized power-mean inequality (2.2), we have, for $p>k\geq 1$,

$$\left[\int_0^{1/2}t\left|t-\frac{1}{2}\right||f''(ta+(1-t)b)|dt\right]^k\leq\left(\int_0^{1/2}t\left|t-\frac{1}{2}\right|dt\right)^{k-\frac{k}{p}}\left(\int_0^{1/2}t\left|t-\frac{1}{2}\right||f''(ta+(1-t)b)|^pdt\right)^{k/p}. \tag{2.8}$$

Similarly, we get

$$\left[\int_{1/2}^1|t-1|\left|t-\frac{1}{2}\right||f''(ta+(1-t)b)|dt\right]^k$$
$$\leq\left(\int_{1/2}^1\left|t-\frac{1}{2}\right||t-1|dt\right)^{k-\frac{k}{p}}\left(\int_{1/2}^1\left|t-\frac{1}{2}\right||t-1||f''(ta+(1-t)b)|^pdt\right)^{k/p}. \tag{2.9}$$

Since $|f''|^p$ is a convex function, we write

$$\int_0^{1/2}t\left|t-\frac{1}{2}\right||f''(ta+(1-t)b)|^pdt\leq\int_0^{1/2}t\left(\frac{1}{2}-t\right)\left[t|f''(a)|^p+(1-t)|f''(b)|^p\right]dt$$

$$\leq \frac{1}{192}\left[\left|f''(a)\right|^p+3\left|f''(b)\right|^p\right].$$

Similarly,

$$\int_{1/2}^{1}\left|t-\frac{1}{2}\right||t-1|\left|f''(ta+(1-t)b)\right|^p dt \leq \int_{1/2}^{1}\left(t-\frac{1}{2}\right)(1-t)\left[t\left|f''(a)\right|^p+(1-t)\left|f''(b)\right|^p\right]dt$$
$$\leq \frac{1}{192}\left[3\left|f''(a)\right|^p+\left|f''(b)\right|^p\right]$$

where

$$\int_0^{1/2} t^2\left(\frac{1}{2}-t\right)dt=\int_{1/2}^{1}\left(t-\frac{1}{2}\right)(1-t)^2dt=\frac{1}{192},$$
$$\int_0^{1/2} t\left(\frac{1}{2}-t\right)dt=\int_{1/2}^{1}\left(t-\frac{1}{2}\right)(1-t)dt=\frac{1}{48},$$
$$\int_0^{1/2}\left(\frac{t}{2}-t^2\right)(1-t)dt=\int_{1/2}^{1}\left(t^2-\frac{t}{2}\right)(1-t)dt=\frac{1}{64}.$$

Combining all obtained inequalities, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^k \leq \frac{(b-a)^{2k}}{2\cdot 48^k}4^{-\frac{k}{p}}\left[\left[\left|f''(a)\right|^p+3\left|f''(b)\right|^p\right]^{k/p}\right.$$
$$\left.+\left[3\left|f''(a)\right|^p+\left|f''(b)\right|^p\right]^{k/p}\right].$$

Using (2.7), we obtain

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^k$$
$$\leq \frac{(b-a)^{2k}}{2.48^k}\left(\frac{3^{k/p}+1}{4^{k/p}}\right)\left[|f"(a)|^k+|f"(b)|^k\right],$$

which completes the proof.

Theorem 2.6. *Let $f: I^0\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I^0 such that $f''\in L_1[a,b]$, where $p\geq 1, k\geq 0$ and $a,b\in I^o$ with $a<b$. If the mapping $|f''|^p$ is concave on $[a,b]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^k \leq \frac{(b-a)^{2k}}{2\cdot 48^k}\left[\left|f''\left(\frac{a+3b}{4}\right)\right|^k+\left|f''\left(\frac{3a+b}{4}\right)\right|^k\right].$$

Proof. Using the concavity of $|f''|^p$ and by the power-mean inequality, we get

$$\left|f''(tx+(1-t)y)\right|^p > t\left|f''(x)\right|^p+(1-t)\left|f''(y)\right|^p \geq \left(t\left|f''(x)\right|+(1-t)\left|f''(y)\right|\right)^p.$$

Hence, we have

$$\left|f''(tx+(1-t)y)\right| \geq t\left|f''(x)\right|+(1-t)\left|f''(y)\right|,$$

so $|f''|$ is also concave. From the inequality (2.3), we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^k \leq \frac{(b-a)^{2k}}{2}\left[\left[\int_0^{1/2} t\left|t-\frac{1}{2}\right|\left|f''(ta+(1-t)b)\right|dt\right]^k\right.$$

$$+\left[\int_{1/2}^{1}\left|t-\frac{1}{2}\right|\left|t-1\right|\left|f''\left(ta+(1-t)b\right)\right|dt\right]^{k}\Bigg].$$

By the Jensen integral inequality, we write

$$\int_{0}^{1/2}t\left|t-\frac{1}{2}\right|\left|f''\left(ta+(1-t)b\right)\right|dt\leq\int_{0}^{1/2}t\left(\frac{1}{2}-t\right)dt$$
$$\times\left|f''\left(\frac{\int_{0}^{1/2}t\left(\frac{1}{2}-t\right)\left(ta+(1-t)b\right)dt}{\int_{0}^{1/2}t\left(\frac{1}{2}-t\right)dt}\right)\right|\leq\frac{1}{48}\left|f''\left(\frac{1}{4}a+\frac{3}{4}b\right)\right|.$$

Similarly,

$$\int_{1/2}^{1}|t-1|\left|t-\frac{1}{2}\right|\left|f''\left(ta+(1-t)b\right)\right|dt$$
$$\leq\left(\int_{1/2}^{1}\left(t-\frac{1}{2}\right)(1-t)dt\right)\left|f''\left(\frac{\int_{1/2}^{1}(1-t)(t-\frac{1}{2})(ta+(1-t)b)dt}{\int_{1/2}^{1}(1-t)(t-\frac{1}{2})dt}\right)\right|$$
$$\leq\frac{1}{48}\left|f''\left(\frac{3}{4}a+\frac{1}{4}b\right)\right|.$$

Combining all obtained inequalities, we get

$$\left|\frac{1}{b-a}\int_{a}^{b}f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^{k}\leq\frac{(b-a)^{2k}}{2\cdot 48^{k}}\left[\left|f''\left(\frac{a+3b}{4}\right)\right|^{k}+\left|f''\left(\frac{3a+b}{4}\right)\right|^{k}\right],$$

which is required.

Corollary 2.7. *i) Under the assumptions of Theorem 2.4 with $k=2$, we get*

$$\left|\frac{1}{b-a}\int_{a}^{b}f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^{2}\leq\frac{(b-a)^{4}}{2}\left(\frac{3}{8}\right)^{2/q}\left(\alpha^{2/p}+\beta^{2/p}\right)\left(\left|f''(a)\right|^{2}+\left|f''(b)\right|^{2}\right).$$

ii) Under the assumptions of Theorem 2.5 with $k=6$, we have

$$\left|\frac{1}{b-a}\int_{a}^{b}f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^{6}\leq\frac{(b-a)^{12}}{2\cdot(48)^{6}}\left(\frac{3^{6/p}+1}{4^{6/p}}\right)\left[\left|f''(a)\right|^{6}+\left|f''(b)\right|^{6}\right].$$

iii) Under the assumptions of Theorem 2.6, we have:
a) When $k=8$, we get

$$\left|\frac{1}{b-a}\int_{a}^{b}f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^{8}\leq\frac{(b-a)^{16}}{2\cdot(48)^{8}}\left[\left|f''\left(\frac{a+3b}{4}\right)\right|^{8}+\left|f''\left(\frac{3a+b}{4}\right)\right|^{8}\right].$$

b) When $M=\sup_{x\in[a,b]}\left|f''(x)\right|$ and $k=1$, we have

$$\left|\frac{1}{b-a}\int_{a}^{b}f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|\leq\frac{(b-a)^{2}}{48}M,$$

which is presented by Liu [11].

Theorem 2.8. *Let $f : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I^0 such that $f'' \in L_1[a,b]$, where $q > k \geq 1$ and $a,b \in I^o$ with $a < b$. If $|f''|^q$ is a s-convex function in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right|^k$$
$$\leq \frac{(b-a)^{2k}}{2}\left[(\alpha^{k/p}\gamma^{k/q} + \beta^{k/p}\rho^{k/q})\left|f''(a)\right|^k + (\alpha^{k/p}\rho^{k/q} + \beta^{k/p}\gamma^{k/q})\left|f''(b)\right|^k\right],$$

where

$$\gamma = \frac{1}{(s+1)2^{s+1}}, \quad \rho = \frac{1}{s+1} - \frac{1}{(s+1)2^{s+1}},$$

and α and β are given by (2.5) and (2.6), respectively.

Proof. Using Lemma 2.3, and by the generalized Hölder's inequality (2.1), we obtain inequality (2.4). Since $|f''|^q$ is a s-convex function in the second sense, we have

$$\int_0^{1/2}\left|f''(ta+(1-t)b)\right|^q dt \leq \int_0^{1/2}\left[t^s\left|f''(a)\right|^q + (1-t)^s\left|f''(b)\right|^q\right]dt$$
$$\leq \frac{1}{(s+1)2^{s+1}}\left|f''(a)\right|^q + \left[\frac{1}{s+1} - \frac{1}{(s+1)2^{s+1}}\right]\left|f''(b)\right|^q. \tag{2.10}$$

Similarly,

$$\int_{1/2}^{1}\left|f''(ta+(1-t)b)\right|^q dt \leq \int_{1/2}^{1}\left[t^s\left|f''(a)\right|^q + (1-t)^s\left|f''(b)\right|^q\right]dt$$
$$\leq \left[\frac{1}{s+1} - \frac{1}{(s+1)2^{s+1}}\right]\left|f''(a)\right|^q + \frac{1}{(s+1)2^{s+1}}\left|f''(b)\right|^q. \tag{2.11}$$

From the inequalities (2.4), (2.5), (2.6), (2.10), and (2.11), we get

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right|^k$$
$$\leq \frac{(b-a)^{2k}}{2}\left[\alpha^{k/p}\left(\gamma\left|f''(a)\right|^q + \rho\left|f''(b)\right|^q\right)^{k/q}\right.$$
$$\left. + \beta^{k/p}\left(\rho\left|f''(a)\right|^q + \gamma\left|f''(b)\right|^q\right)^{k/q}\right],$$

where

$$\gamma = \int_0^{1/2} t^s dt = \int_{1/2}^{1}(1-t)^s dt = \frac{1}{(s+1)2^{s+1}} \tag{2.12}$$

$$\rho = \int_0^{1/2}(1-t)^s dt = \int_{1/2}^{1} t^s dt = \frac{1}{s+1} - \frac{1}{(s+1)2^{s+1}}. \tag{2.13}$$

Using the inequality (2.7), we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right|^k$$

$$\leq \frac{(b-a)^{2k}}{2}\Bigg[(\alpha^{k/p}\gamma^{k/q}+\beta^{k/p}\rho^{k/q})\left|f''(a)\right|^k$$
$$+(\alpha^{k/p}\rho^{k/q}+\beta^{k/p}\gamma^{k/q})\left|f''(b)\right|^k\Bigg].$$

which completes the proof.

Corollary 2.9. *Under the assumptions of Theorem 2.8 with* $s=1$, $k=12$ *and* $M=\sup_{x\in[a,b]}|f''(x)|$, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^{12}\leq\frac{(b-a)^{24}}{2}\Bigg[\left(\frac{\alpha^{12/p}+3\beta^{12/p}}{8}\right)\left|f''(a)\right|^{12}$$
$$+\left(\frac{3\alpha^{12/p}+\beta^{12/p}}{8}\right)\left|f''(b)\right|^{12}\Bigg]$$
$$\leq\frac{(b-a)^{24}}{4}\left(\alpha^{12/p}+\beta^{12/p}\right)M^{12}.$$

Theorem 2.10. *Let* $f:I^0\subset\mathbb{R}\to\mathbb{R}$ *be a twice differentiable mapping on* I^0 *such that* $f''\in L_1[a,b]$, *where* $q>k\geq 1$ *and* $a,b\in I^o$ *with* $a<b$. *If* $|f''|^q$ *is a s-convex function in the second sense on* $[a,b]$ *for some fixed* $s\in(0,1)$, *then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^{k}\leq\frac{(b-a)^{2k}}{2}\left(\frac{1}{s+1}\right)^{k/q}\Bigg[\beta^{k/p}\left|f''(a)\right|^k+\alpha^{k/p}\left|f''(b)\right|^k$$
$$+(\alpha^{k/p}+\beta^{k/p})\left|f''\left(\frac{a+b}{2}\right)\right|^k\Bigg]$$
$$\leq\frac{(b-a)^{2k}}{2}\Bigg[\beta^{k/p}\left|f''(a)\right|^k+\alpha^{k/p}\left|f''(b)\right|^k$$
$$+(\alpha^{k/p}+\beta^{k/p})\left|f''\left(\frac{a+b}{2}\right)\right|^k\Bigg].\tag{2.14}$$

where α *and* β *are given by* (2.5) *and* (2.6), *respectively.*

Proof. Using Lemma 2.3, and by the generalized Hölder's inequality (2.1), we obtain inequality (2.4). Since $|f''|^q$ is a s-convex function in the second sense, using (1.2), we have

$$\int_0^{1/2}\left|f''(ta+(1-t)b)\right|^q dt\leq\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q}{s+1}\tag{2.15}$$

and

$$\int_{1/2}^1\left|f''(ta+(1-t)b)\right|^q dt\leq\frac{|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q}{s+1}.\tag{2.16}$$

From the inequalities (2.4), (2.5), (2.6), (2.15), and (2.16), we obtain

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^{k}\leq\frac{(b-a)^{2k}}{2}\left(\frac{1}{s+1}\right)^{k/q}$$

$$\times\left[\alpha^{k/p}\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q\right)^{k/q}+\beta^{k/p}\left(|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{k/q}\right].$$

Using (2.7), we obtain

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^k\leq\frac{(b-a)^{2k}}{2}\left(\frac{1}{s+1}\right)^{k/q}\times\left[\beta^{k/p}|f''(a)|^k+\alpha^{k/p}|f''(b)|^k+(\alpha^{k/p}+\beta^{k/p})\left|f''\left(\frac{a+b}{2}\right)\right|^k\right],$$

which completes the proof of the first inequality in (2.14). The second inequality in (2.14) follows from the fact that $\left(\frac{1}{s+1}\right)^{k/q}\leq 1$ for $s\in(0,1)$ and $q>1$.

Corollary 2.11. *Under the assumptions of Theorem 2.10 with* $M=\sup_{x\in[a,b]}|f''(x)|$ *and* $k=24$*, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^{24}\leq(b-a)^{48}\left(\alpha^{\frac{24}{p}}+\beta^{\frac{24}{p}}\right)M^{24}.$$

Theorem 2.12. *Let* $f:I^0\subset\mathbb{R}\to\mathbb{R}$ *be a twice differentiable mapping on* I^0 *such that* $f''\in L_1[a,b]$*, where* $q>k\geq 1$ *and* $a,b\in I^o$ *with* $a<b$*. If* $|f''|^q$ *is a s-concave function on* $[a,b]$ *for some fixed* $s\in(0,1)$*, then we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^k\\&\leq\frac{(b-a)^{2k}}{2}2^{\frac{k(s-1)}{q}}\left[\alpha^{k/p}\left|f''\left(\frac{a+3b}{4}\right)\right|^k+\beta^{k/p}\left|f''\left(\frac{3a+b}{4}\right)\right|^k\right]\\&\leq\frac{(b-a)^{2k}}{2}\left[\alpha^{k/p}\left|f''\left(\frac{a+3b}{4}\right)\right|^k+\beta^{k/p}\left|f''\left(\frac{3a+b}{4}\right)\right|^k\right],\end{aligned}\tag{2.17}$$

where α *and* β *are given by* (2.5) *and* (2.6)*, respectively.*

Proof. Similarly as in Theorem 2.10, since $|f''|^q$ is a s-concave, we use again (1.2), but now

$$\int_0^{1/2}|f''(ta+(1-t)b)|^q dt\leq 2^{s-1}\left|f''\left(\frac{a+3b}{4}\right)\right|^q,$$

$$\int_{1/2}^1|f''(ta+(1-t)b)|^q dt\leq 2^{s-1}\left|f''\left(\frac{3a+b}{4}\right)\right|^q.$$

Thus, the first inequality in (2.17) follows. The second inequality in (2.17) follows from the fact $2^{\frac{k(s-1)}{q}}\leq 1$, for $s\in(0,1)$ and $q>1$.

Corollary 2.13. *Under the assumptions of Theorem 2.12 with*

$$M=\sup_{x\in[a,b]}|f''(x)|$$

and $k=25$*, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^{25}\leq\frac{(b-a)^{50}}{2}\left(\alpha^{\frac{25}{p}}+\beta^{\frac{25}{p}}\right)M^{25}.$$

Theorem 2.14. *Let $f : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I^0 such that $f'' \in L_1[a,b]$, where $q > k \geq 1$ and $a,b \in I^o$ with $a < b$. If $|f''|^q$ is a s-convex function in the first sense on $[a,b]$ for some fixed $s \in (0,1]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right|^k \leq \frac{(b-a)^{2k}}{2}\left[(\alpha^{k/p}\gamma^{k/q} + \beta^{k/p}\rho^{k/q})\left|f''(a)\right|^k\right.$$
$$\left. + (\alpha^{k/p}(\frac{1}{2}-\gamma)^{k/q} + \beta^{k/p}(\frac{1}{2}-\rho)^{k/q})\left|f''(b)\right|^k\right].$$

where α, β, γ and ρ are given by (2.5), (2.6), (2.12) and (2.13), respectively.

Proof. Using Lemma 2.3, and by the generalized Hölder's inequality (2.1), we have inequality (2.4). Since $|f''|^q$ is a s-convex function in the first sense on $[a,b]$, we obtain

$$\int_0^{1/2}\left|f''(ta+(1-t)b)\right|^q dt \leq \int_0^{1/2}\left[t^s\left|f''(a)\right|^q + (1-t^s)\left|f''(b)\right|^q\right]dt$$
$$\leq \frac{1}{(s+1)2^{s+1}}\left|f''(a)\right|^q + \left[\frac{1}{2} - \frac{1}{(s+1)2^{s+1}}\right]\left|f''(b)\right|^q. \tag{2.18}$$

Similarly,

$$\int_{1/2}^{1}\left|f''(ta+(1-t)b)\right|^q dt \leq \int_{1/2}^{1}\left[t^s\left|f''(a)\right|^q + (1-t^s)\left|f''(b)\right|^q\right]dt$$
$$\leq \left[\frac{1}{s+1} - \frac{1}{(s+1)2^{s+1}}\right]\left|f''(a)\right|^q$$
$$+ \left[\frac{1}{2} - \left(\frac{1}{s+1} - \frac{1}{(s+1)2^{s+1}}\right)\right]\left|f''(b)\right|^q. \tag{2.19}$$

From the inequalities (2.4), (2.5), (2.6), (2.12), (2.13), (2.18) and (2.19), we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right|^k$$
$$\leq \frac{(b-a)^{2k}}{2}\left[\alpha^{k/p}\left(\gamma|f''(a)|^q + \left(\frac{1}{2}-\gamma\right)|f''(b)|^q\right)^{k/q}\right.$$
$$\left. + \beta^{k/p}\left(\rho|f''(a)|^q + \left(\frac{1}{2}-\rho\right)|f''(b)|^q\right)^{k/q}\right].$$

Using the inequality (2.7), we get

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right|^k$$
$$\leq \frac{(b-a)^{2k}}{2}\left[(\alpha^{k/p}\gamma^{k/q} + \beta^{k/p}\rho^{k/q})\left|f''(a)\right|^k\right.$$
$$\left. + (\alpha^{k/p}(\frac{1}{2}-\gamma)^{k/q} + \beta^{k/p}(\frac{1}{2}-\rho)^{k/q})\left|f''(b)\right|^k\right]$$

which completes the proof.

Theorem 2.15. *Let $f : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I^0 such that $f'' \in L_1[a,b]$, where $p > k \geq 1$ and $a,b \in I^o$ with $a < b$. If $|f''|^p$ is a s-convex function in the first sense on $[a,b]$ for some fixed $s \in (0,1]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right|^k$$
$$\leq \frac{(b-a)^{2k}}{2\cdot 48^{k(1-1/p)}}\left[(\eta^{k/p}+\mu^{k/p})\left|f''(a)\right|^k\right.$$
$$\left. + \left(\left(\frac{1}{48}-\eta\right)^{k/p} + \left(\frac{1}{48}-\mu\right)^{k/p}\right)\left|f''(b)\right|^k\right].$$

Proof. Using Lemma 2.3, and by the generalized power-mean inequality (2.2), we have inequalities (2.8) and (2.9). Since $|f''|^p$ is a s-convex function in the first sense, we obtain

$$\int_0^{1/2} t\left|t-\frac{1}{2}\right|\left|f''(ta+(1-t)b)\right|^p dt \leq \int_0^{1/2} t\left(\frac{1}{2}-t\right)\left[t^s\left|f''(a)\right|^p + (1-t^s)\left|f''(b)\right|^p\right] dt$$
$$\leq \eta\left|f''(a)\right|^p + \left(\frac{1}{48}-\eta\right)\left|f''(b)\right|^p. \tag{2.20}$$

Similarly,

$$\int_{1/2}^1 \left|t-\frac{1}{2}\right||t-1|\left|f''(ta+(1-t)b)\right|^p dt \leq \int_{1/2}^1 \left(t-\frac{1}{2}\right)(1-t)\left[t^s\left|f''(a)\right|^p + (1-t^s)\left|f''(b)\right|^p\right] dt.$$
$$\leq \mu\left|f''(a)\right|^p + \left(\frac{1}{48}-\mu\right)\left|f''(b)\right|^p. \tag{2.21}$$

where,

$$\eta = \int_0^{1/2} t^{s+1}\left(\frac{1}{2}-t\right)dt = \frac{1}{2^{s+3}(s+2)(s+3)},$$
$$\int_0^{1/2} t\left(\frac{1}{2}-t\right)(1-t^s)\,dt = \frac{1}{48}-\eta,$$
$$\mu = \int_{1/2}^1 \left(t-\frac{1}{2}\right)(1-t)t^s dt = \int_1^{\frac{1}{2}}\left(t^{s+2}-\frac{3}{2}t^{s+1}+\frac{1}{2}t^s\right)dt$$
$$= \frac{1-2^{s+3}}{2^{s+3}(s+3)} - \frac{3(1-2^{s+2})}{2^{s+3}(s+2)} + \frac{1-2^{s+1}}{2^{s+2}(s+1)}.$$
$$\int_{1/2}^1 \left(t-\frac{1}{2}\right)(1-t)(1-t^s)\,dt = \frac{1}{48}-\mu,$$
$$\int_0^{1/2} t\left|t-\frac{1}{2}\right|dt = \int_{1/2}^1 \left|t-\frac{1}{2}\right||t-1|\,dt = \frac{1}{48}.$$

From the inequalities (2.8), (2.9), (2.20), and (2.21), we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right|^k$$
$$\leq \frac{(b-a)^{2k}}{2\cdot 48^k}\left(\frac{1}{48}\right)^{-k/p}\left[\eta\left[\left|f''(a)\right|^p + \left(\frac{1}{48}-\eta\right)\left|f''(b)\right|^p\right]^{k/p}\right.$$

$$+\left[\mu\left|f''(a)\right|^p+\left(\frac{1}{48}-\mu\right)\left|f''(b)\right|^p\right]^{k/p}\Bigg].$$

Using (2.7), we get

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^k$$
$$\leq\frac{(b-a)^{2k}}{2\cdot 48^k}\left(\frac{1}{48}\right)^{-k/p}\left[(\eta^{k/p}+\mu^{k/p})\left|f''(a)\right|^k+((\frac{1}{48}-\eta)^{k/p}+(\frac{1}{48}-\mu)^{k/p})\left|f''(b)\right|^k\right],$$

which is required.

Corollary 2.16. *Under the assumptions of Theorem 2.5 and Theorem 2.15, the following hold:*
(i) When $k=1$, $s=1$, and $p=2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|\leq(b-a)^2\left(\frac{\sqrt{3}+1}{192}\right)\left[\left|f''(a)\right|+\left|f''(b)\right|\right].$$

(ii) When $s=1$, $k=2$, and $p=4$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^2\leq(b-a)^4\left(\frac{\sqrt{3}+1}{96^2}\right)\left[\left|f''(a)\right|^2+\left|f''(b)\right|^2\right].$$

(iii) When $s=1$, $k=n$, and $p=2n$, for $n\geq 1$, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|^n$$
$$\leq(b-a)^{2n}\left(\frac{\sqrt{3}+1}{48^{n-2}\cdot 96^2}\right)\left[\left|f''(a)\right|^n+\left|f''(b)\right|^n\right].$$

where

$$\frac{\sqrt{3}+1}{192}=0.0142294313\ldots,$$
$$\frac{1}{96}=0.0104166667\ldots.$$

This value is very close to $\frac{1}{96}$.

3. Applications to Special Means and to Polygamma Functions

We shall consider the means for arbitrary real numbers α,β. We take

$$A(\alpha,\beta)=\frac{\alpha+\beta}{2},\qquad\text{(arithmetic mean)}$$
$$L(\alpha,\beta)=\frac{\beta-\alpha}{\ln|\beta|-\ln|\alpha|},\qquad |\alpha|\neq|\beta|,\quad \alpha\beta\neq 0,\qquad\text{(logarithmic mean)}$$
$$Ln(\alpha,\beta)=\left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{1/n},\qquad n\in\mathbb{Z}\setminus\{-1,0\},\quad \alpha,\beta\in\mathbb{R},\quad \alpha\neq\beta,\qquad\text{(generalized log-mean)}$$

Now, using the results of Section 2, we give some applications to special means of real numbers and to polygamma functions $\psi^{(m)}(x)$ for $m=0,1,2,3$.

In [1], the following example is given:
Let $s \in (0,1)$ and $a,b,c \in \mathbb{R}$. We define the function $f:[0,\infty) \to \mathbb{R}$ as

$$f(t) = \begin{cases} a, & t = 0 \\ bt^s + c, & t > 0 \end{cases}$$

If $b \geq 0$ and $0 \leq c \leq a$, then $f \in K_s^2$. Hence, for $a = c = 0, b = 1$, we have $f(t) = t^s$, $f:[0,1] \to [0,1]$, and $f \in K_s^2$.
Proposition 3.1. *Let* $a,b \in I^o$, $a < b$, $0 \notin [a,b]$, *and*

$$\left| L_{-2n}^{-2n}(a,b) - \frac{1}{2}\left[A\left(a^{-2n}, b^{-2n}\right) + A^{-2n}(a,b)\right]\right|^k$$
$$\leq 2\left|n(2n+1)\right|(b-a)^{2k}\left(\frac{3}{8}\right)^{k/q}(\alpha^{k/p}+\beta^{k/p})A(|a|^{-k(2n+2)}, |b|^{-k(2n+2)}).$$

Proof. The assertion follows from Theorem 2.4 applied to the convex function

$$f(x) = x^{-2n}, \quad x \in \mathbb{R}, \quad n \in \mathbb{Z}^-.$$

Proposition 3.2. *Let* $a,b \in [0,\infty)$, $a < b$. *Then we have, for all* $q > 1$

$$\left|\frac{b_1 - a_1}{b-a} - \frac{1}{2}(A(a_2,b_2) + \cosh A(a,b))\right|^k \leq (b-a)^{2k}\left(\frac{3}{8}\right)^{k/q}(\alpha^{k/p}+\beta^{k/p})A(|a_2|^k, |b_2|^k),$$

where $a_1 = \sinh a$, $b_1 = \sinh b$, $a_2 = \cosh a$, $b_2 = \cosh b$.
Proof. The assertion follows from Theorem 2.4 applied to the convex function

$$f(x) = \cosh x, \quad f:[0,\infty) \to \mathbb{R}^+.$$

Proposition 3.3. *Let* $a,b \in (0,\infty)$, $a < b$. *Then we have*

$$\left|\ln\left[\frac{I^{(b-a)}(a+1,b+1)}{\sqrt{G(a+1,b+1)A(a+1,b+1)}}\right]\right|^k \leq \frac{(b-a)^{2k}(3^{k/p}+1)}{48^k \cdot 4^{k/p}}A\left(\frac{1}{(a+1)^{2k}}, \frac{1}{(b+1)^{2k}}\right).$$

Proof. The assertion follows from Theorem 2.5 applied to the convex function

$$f(x) = -\ln(x+1), \quad f:(0,\infty) \to \mathbb{R}.$$

Proposition 3.4. *Let* $a,b \in [0,\infty)$, $a < b$. *Then we have:*
i.

$$\left|L(e^a,e^b) - \frac{1}{2}(A(e^a,e^b) + e^{A(a,b)})\right|^k \leq \frac{(b-a)^{2k}(3^{k/p}+1)}{48^k \cdot 4^{k/p}}A(e^{ak}, e^{bk}).$$

ii.

$$\left|L(e^a,e^b) - \frac{1}{2}(A(e^a,e^b) + e^{A(a,b)})\right|^n \leq \frac{(b-a)^{2n}(\sqrt{3}+1)}{48^{n-1} \cdot 96}A(e^{na}, e^{nb}).$$

Proof. The assertions follow from Theorem 2.5 and Corollary 2.16-iii applied to the convex function

$$f(x) = e^x, \quad f:[0,\infty) \to \mathbb{R}.$$

Proposition 3.5. *Let* $a,b \in I^o$, $a < b$, $0 \notin [a,b]$, *and* $0 < n < 1$. *Then we have:*

$$\left|L_n^n(a,b) - \frac{1}{2}\left[A(a^n,b^n) + A^n(a,b)\right]\right|^k \leq n(n-1)\frac{(b-a)^{2k}}{48^k}A\left(\left|\frac{a+3b}{4}\right|^{k(n-2)}, \left|\frac{3a+b}{4}\right|^{k(n-2)}\right).$$

Proof. The assertion follows from Theorem 2.6 applied to the concave function

$$f(x)=x^n,\quad x\in[a,b],\quad 0<n<1.$$

Proposition 3.6. *Let* $a,b\in I^o$, $a<b$, *and* $0<s<1$. *Then we have, for all* $q>1$:

$$\left|L_s^s(a,b)-\frac{1}{2}\left[A\left(a^s,b^s\right)+A^s(a,b)\right]\right|^k$$
$$\leq s(s-1)\frac{(b-a)^{2k}}{2}\Big[\left(\alpha^{k/p}\gamma^{k/q}+\beta^{k/p}\rho^{k/q}\right)|a|^{k(s-2)}$$
$$+\left(\alpha^{k/p}\rho^{k/q}+\beta^{k/p}\gamma^{k/q}\right)|b|^{k(s-2)}\Big].$$

Proof. The assertion follows from Theorem 2.8 applied to the s-convex function

$$f(x)=x^s,\quad f:[0,1]\to[0,1].$$

Proposition 3.7. *Let* $a,b\in I^o$, $a<b$ *and* $0<s<1$. *Then we have, for all* $q>1$:

$$\left|L_s^s(a,b)-\frac{1}{2}\left[A\left(a^s,b^s\right)+A(a,b)^s\right]\right|^k\leq s(s-1)\frac{(b-a)^{2k}}{2}\Bigg[\beta^{k/p}\left(|a|\right)^{k(s-2)}$$
$$+\alpha^{k/p}\left(|b|\right)^{k(s-2)}+\left(\alpha^{k/p}+\beta^{k/p}\right)\left(\frac{|a+b|}{2}\right)^{k(s-2)}\Bigg].$$

Proof. The assertion follows from Theorem 2.10 applied to the s-convex function

$$f(x)=x^s,\quad f:[0,1]\to[0,1].$$

In [18], the polygamma functions $\psi^{(m)}(x)$ of order m, $m\in\mathbb{N}_0$, are defined by $\psi^{(0)}(x):=\psi(x)$ and $\psi^{(n)}(x):=\frac{d^n}{dx^n}\psi(x)$, where $n\in\mathbb{N}:=\{1,2,3,\dots\}$; $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, $x\notin\mathbb{Z}_0^-:=\{0,-1,-2,\dots\}$, and $\psi(x)$ is the psi (or digamma) function, given as the logarithmic derivative of the familiar Gamma function $\Gamma(x)$:

$$\psi(x)=\frac{d}{dx}\log\Gamma(x)=\frac{\Gamma'(x)}{\Gamma(x)}.$$

We can give the following two propositions for the polygamma functions. Here, $\psi^{(0)}(x)$ and $\psi^{(2)}(x)$ are concave functions, while $\psi^{(1)}(x)$ and $\psi^{(3)}(x)$ are convex functions for $x\in(0,\infty)$.

Proposition 3.8. *Let* $a,b\in I^o$, $a<b$, $k\geq 0$, *and* $i=0,2$. *Then we have*

$$\left|\frac{\psi^{(i-1)}(b)-\psi^{(i-1)}(a)}{b-a}-\frac{1}{2}\left[\frac{\psi^{(i)}(a)+\psi^{(i)}(b)}{2}+\psi^{(i)}\left(\frac{a+b}{2}\right)\right]\right|^k$$
$$\leq\frac{(b-a)^{2k}}{2.48^k}\left[\left|\psi^{(i+2)}\left(\frac{a+3b}{4}\right)\right|^k+\left|\psi^{(i+2)}\left(\frac{3a+b}{4}\right)\right|^k\right].$$

Proof. The assertion follows from Theorem 2.6 applied to the concave functions $\psi^{(i)}(x), i=0,2$, for $x\in(0,\infty)$, where $\psi^{(-1)}(x)=\log\Gamma(x)$.

Proposition 3.9. *Let* $a,b\in I^o$, $a<b$, $p>k\geq 1$, *and* $i=1,3$. *Then we have*

$$\left|\frac{\psi^{(i-1)}(b)-\psi^{(i-1)}(a)}{b-a}-\frac{1}{2}\left[\frac{\psi^{(i)}(a)+\psi^{(i)}(b)}{2}+\psi^{(i)}\left(\frac{a+b}{2}\right)\right]\right|^k$$
$$\leq\frac{(b-a)^{2k}(3^{k/p}+1)}{2.48^k\cdot 4^{k/p}}\left[\left|\psi^{(i+2)}\left(\frac{a+3b}{4}\right)\right|^k+\left|\psi^{(i+2)}\left(\frac{3a+b}{4}\right)\right|^k\right].$$

Proof. The assertion follows from Theorem 2.5 applied to the convex functions $\psi^{(i)}(x), i=1,3$, for $x\in(0,\infty)$.

4. Conclusion

In this paper, we presented a generalization of the power-mean inequality and derived several Bullen-type inequalities with the kth power for twice differentiable mappings using s-convex functions in both the first and second senses, as well as s-concave functions. Furthermore, we provided applications to special means of real numbers and to polygamma functions $\psi^{(m)}(x)$ for $m = 0, 1, 2, 3$.

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