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Research Article

On the Schur m-Power Convexity of the Generalized Invariant Contra-Harmonic Means

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Abstract

In this paper, we investigate the Schur m-power convexity of the generalized invariant contra harmonic means, then some related results are generalized. As applications, some new inequalities involving Gini mean, arithmetic mean, geometric mean and harmonic mean are established.

Keywords: Schur m-power convexity, invariant contra harmonic means, convex functions

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1. Introduction

As we all know, inequalities are an important mathematical tool for studying natural sciences. Convex functions play a very important role in constructing and proving inequalities. Schur-convex functions are a generalized convex functions and are often used to establish and prove many important symmetric inequalities [1].

In 2017, K. R. Sampath et al. defined the inverse contra harmonic mean as follows([2]):

$$V(a,b)=\frac{H(a^2,b^2)}{H(a,b)}=\frac{ab(a+b)}{a^2+b^2},$$

where $a,b\in\mathbb{R}_+=(0,+\infty)$ and $H(a,b)=\frac{2}{\frac{1}{a}+\frac{1}{b}}=\frac{2ab}{a+b}$ is the harmonic mean. Moreover, they established the following results.

Theorem 1.1 ([2, Theorem 2.1])**.** *Let $a,b\in\mathbb{R}_+$, then*
(1) *$V(a,b)$ is Schur m-power convex if $-\frac{2}{3}<m<\frac{1}{2}$;*
(2) *$V(a,b)$ is Schur m-power concave if $m\in\left(-\infty-\frac{2}{3}\right)\cup\left(\frac{1}{2},+\infty\right)$.*

In 2024, Shi et al. defined the generalized invariant contra harmonic means $V_k(a,b)$ and $V_{p,q}(a,b)$ in [3] as follows:

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Definition 1.2 ([3, Deffnition 5]). Let $a,b \in \mathbb{R}_+$ and $p,q,k \in \mathbb{N}$ with $p \le q$. Define

$$V_{p,q}(a,b) = \frac{H(a^q,b^q)}{H(a^p,b^p)} \tag{1.1}$$

and

$$V_k(a,b) = V_{k,k+1}(a,b),$$

where $H(a,b)$ is the harmonic mean.

Moreover, they established the following results.

Theorem 1.3 ([3, Theorem 4]). *Let $a,b \in \mathbb{R}_+$ and $k \in \mathbb{N}$.*
(a) If $0 \le m \le 1$, then $V_k(a,b)$ is Schur m-power concave with $(a,b) \in \mathbb{R}^2_+$;
(b) $V_k(a,b)$ is Schur-geometrically concave with $(a,b) \in \mathbb{R}^2_+$;
(c) $V_k(a,b)$ is Schur-harmonically concave with $(a,b) \in \mathbb{R}^2_+$.

Theorem 1.4 ([3, Corollary 6]). *Let $(a,b) \in \mathbb{R}^2_+$ and $p,q \in \mathbb{N}$ with $p < q$.*
(a) If $0 \le m \le 1$, then $V_{p,q}(a,b)$ is Schur m-power concave with $(a,b) \in \mathbb{R}^2_+$;
(b) $V_{p,q}(a,b)$ is Schur-geometrically concave with $(a,b) \in \mathbb{R}^2_+$;
(c) $V_{p,q}(a,b)$ is Schur-harmonically concave with $(a,b) \in \mathbb{R}^2_+$.

In this paper, we generalize the condition of $p,q \in \mathbb{N}$ with $p \le q$ in Definition 1.2 to $p,q \in \mathbb{R}$ with $q \ne p$, and give the following definition.

Definition 1.5. For $a,b \in \mathbb{R}_+$ and $p,q \in \mathbb{R}$ with $q \ne p$. Define generalized invariant contra harmonic means

$$V_{p,q}(a,b) = \frac{H(a^q,b^q)}{H(a^p,b^p)}, \tag{1.2}$$

where $H(a,b)$ is the harmonic mean.

Moreover, we discuss the Schur m-power convexity of $V_{p,q}(a,b)$ for $p,q \in \mathbb{R}$ with $p \ne q$ and $m \in \mathbb{R}$, then Theorems 1.3 and 1.4 are generalized. Furthermore, we establish some new inequalities by using the Schur m-power convexity of $V_{p,q}(a,b)$ and the theory of majorization.

2. Schur m-power convexity and some lemmas

In order to verify our main results, the following definitions are necessary.

Definition 2.1 ([1, 4]). Let $\boldsymbol{x} := (x_1,\ldots,x_n), \boldsymbol{y} := (y_1,\ldots,y_n) \in \mathbb{R}^n$.

(1) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}, k = 1,2,\ldots,n-1 \quad \text{and} \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i,$$

where $x_{[1]} \ge \cdots \ge x_{[n]}$ and $y_{[1]} \ge \cdots \ge y_{[n]}$ are rearrangements of x and y in a descending order.

(2) Let $\Omega \subseteq \mathbb{R}^n$. The function φ: $\Omega \to \mathbb{R}$ is said to be a Schur convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$. φ is said to be a Schur concave function on Ω if and only if $-\varphi$ is a Schur convex function.

Definition 2.2 ([5–7]). Let $f : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{x^m - 1}{m}, & m \ne 0; \\ \ln x, & m = 0. \end{cases} \tag{2.1}$$

The function $\varphi : \Omega \subseteq \mathbb{R}^n_+ \to \mathbb{R}$ is said to be Schur m-power convex on Ω if

$$f(\boldsymbol{x}) := (f(x_1),\ldots,f(x_n)) \prec f(\boldsymbol{y}) := (f(y_1),\ldots,f(y_n)) \tag{2.2}$$

on Ω implies that $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$.

Remark 2.3. Putting $f(x)=x$; $\ln x$; x^{-1} in Definition 2.2 yield the Schur-convexity (see [1, 4]), Schur-geometrical convexity (see [8, 9]) and Schur-harmonic convexity (see [10–12]), respectively.

In the papers [13–22], for example, there have been many results on investigations of the Schur m-convexity and related ones.

Now, we introduce several lemmas.

Lemma 2.4 ([5–7])**.** *Let $\Omega \subseteq \mathbb{R}_+^n$ be a symmetric set with nonempty interior Ω° and $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω°. Then φ is Schur m-power convex on Ω if and only if φ is symmetric on Ω and*

$$\frac{x_1^m - x_2^m}{m}\left(x_1^{1-m}\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1} - x_2^{1-m}\frac{\partial\varphi(\boldsymbol{x})}{\partial x_2}\right) \ge 0, \text{ if } m \ne 0, \boldsymbol{x} \in \Omega^\circ \tag{2.3}$$

and

$$(\ln x_1 - \ln x_2)\left(x_1\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1} - x_2\frac{\partial\varphi(\boldsymbol{x})}{\partial x_2}\right) \ge 0, \text{ if } m = 0, \boldsymbol{x} \in \Omega^\circ. \tag{2.4}$$

Remark 2.5. Letting $m = 1, 0, -1$ in Lemma 2.4 respectively, we can deduce the criteria theorems for the Schur-convexity (see [1, 4]), Schur-geometrical convexity (see [8, 9]), and Schur-harmonic convexity (see [10–12]) respectively.

Let $a, b \in \mathbb{R}_+$ and $p, q \in \mathbb{R}$. The Gini mean [23] are defined as

$$G_{p,q}(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{a^q + b^q}\right)^{\frac{1}{p-q}}, & p \ne q, \\ \exp\left(\dfrac{a^p \ln a + b^p \ln b}{a^p + b^p}\right), & p = q. \end{cases} \tag{2.5}$$

Yang established the following necessary and sufficient conditions for the Schur m-convexity of Gini mean in [7]:

Lemma 2.6 ([7, Theorem 1.1 and 1.2])**.** *For $m > 0$ and fixed $(p,q) \in \mathbb{R}^2$, Gini mean $G_{p,q}(a,b)$ is Schur m-power convex (Schur m-power concave) with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $p + q \ge (\le) m$ and $\min\{p,q\} \ge (\le) 0$.*

Lemma 2.7 ([7, Theorem 1.3 and 1.4])**.** *For $m < 0$ and fixed $(p,q) \in \mathbb{R}^2$, Gini mean $G_{p,q}(a,b)$ is Schur m-power convex (Schur m-power concave) with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $p + q \ge (\le) m$ and $\max\{p,q\} \ge (\le) 0$.*

Lemma 2.8 ([7, Theorem 1.5])**.** *For $m = 0$ and fixed $(p,q) \in \mathbb{R}^2$, Gini mean $G_{p,q}(a,b)$ is Schur-geometrically convex (Schur-geometrically concave) with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $p + q \ge (\le) 0$.*

3. Schur m-power convexity of $V_{p,q}(a,b)$

Let $a, b \in \mathbb{R}_+$ and $p, q \in \mathbb{R}$ with $q \ne p$. By the definition of generalized invariant contra harmonic mean (1.2) and the definition of Gini mean (2.5), we have

$$V_{p,q}(a,b) = \frac{H(a^q, b^q)}{H(a^p, b^p)} = \frac{a^{-p} + b^{-p}}{a^{-q} + b^{-q}} = \left[G_{\overline{p},\overline{q}}(a,b)\right]^{q-p}, \tag{3.1}$$

where $\overline{q} := -q$ and $\overline{p} = -p$. Therefore, we first investigate the Schur m-power convexity relationship between $V_{p,q}(a,b)$ and $G_{\overline{p},\overline{q}}(a,b)$.

Theorem 3.1. *For $m \in \mathbb{R}$ and $(p,q) \in \mathbb{R}^2$ with $p \ne q$. Then the Schur m-power convexity of $V_{p,q}(a,b)$ is equivalent to the Schur m-power convexity of the $(q-p)G_{\overline{p},\overline{q}}(a,b)$ on $(a,b) \in \mathbb{R}_+^2$, where $G_{\overline{p},\overline{q}}(a,b)$ is defined by (3.1).*

Proof. For any $m \in \mathbb{R}$ with $m \neq 0$ $a,b \in \mathbb{R}_+$, it is easy to deduce

$$\frac{a^m - b^m}{m}(a-b) \geq 0 \quad \text{and} \quad (\ln a - \ln b)(a-b) \geq 0.$$

For $(a,b) \in \mathbb{R}^2_+$ and $m \in \mathbb{R}$, by the (2.3) and (2.4) in Lemma 2.4, we obtain

$$\begin{aligned}&(a-b)\left(a^{1-m}\frac{\partial V_{p,q}(a,b)}{\partial a} - b^{1-m}\frac{\partial V_{p,q}(a,b)}{\partial b}\right)\\ =&(q-p)(a-b)\left(a^{1-m}\frac{\partial G_{\overline{p},\overline{q}}(a,b)}{\partial a} - b^{1-m}\frac{\partial G_{\overline{p},\overline{q}}(a,b)}{\partial b}\right)\left[G_{\overline{p},\overline{q}}(a,b)\right]^{q-p-1}.\end{aligned}$$

So, the Schur m-power convexity of $V_{p,q}(a,b)$ and $(q-p)G_{\overline{p},\overline{q}}(a,b)$ are equivalent. Theorem 3.1 is thus proved. □

Now, we use Theorem 3.1 and Lemma 2.6 to prove the following Theorems 3.2.

Theorem 3.2. *For $m > 0$ and some fixed $(p,q) \in \mathbb{R}^2$ with $p \neq q$.*

(1) If $q > p$, then $V_{p,q}(a,b)$ is Schur m-power convex (Schur m-power concave) with respect to $(a,b) \in \mathbb{R}^2_+$ if and only if $p+q \leq (\geq) -m$ and $q \leq (\geq) 0$;

(2) If $q < p$, then $V_{p,q}(a,b)$ is Schur m-power convex (Schur m-power concave) with respect to $(a,b) \in \mathbb{R}^2_+$ if and only if $p+q \geq (\leq) -m$ and $p \geq (\leq) 0$.

Proof. If $q > p$, by Theorem 3.1 and Lemma 2.6, $V_{p,q}(a,b)$ is Schur m-power convex (Schur m-power concave) with respect to $(a,b) \in \mathbb{R}^2_+$ if and only if

$$\overline{p} + \overline{q} = -(p+q) \geq (\leq) m \quad \text{and} \quad \min\{\overline{p},\overline{q}\} = \min\{-p,-q\} \geq (\leq) 0.$$

Therefore (1) is true. A similar discussion leads to (2). The proof of Theorem 3.2 is completed. □

Similarly, we can use Theorem 3.1 and Lemmas 2.7 and 2.8 to prove the following Theorems 3.3 and 3.4.

Theorem 3.3. *For $m < 0$ and some fixed $(p,q) \in \mathbb{R}^2$ with $p \neq q$.*

(1) If $q > p$, then $V_{p,q}(a,b)$ is Schur m-power convex (Schur m-power concave) with respect to $(a,b) \in \mathbb{R}^2_+$ if and only if $p+q \leq (\geq) -m$ and $p \leq (\geq) 0$;

(2) If $q < p$, then $V_{p,q}(a,b)$ is Schur m-power convex (Schur m-power concave) with respect to $(a,b) \in \mathbb{R}^2_+$ if and only if $p+q \geq (\leq) -m$ and $q \geq (\leq) 0$.

Theorem 3.4. *For $m = 0$ and some fixed $(p,q) \in \mathbb{R}^2$ with $p \neq q$.*

(1) If $q > p$, then $V_{p,q}(a,b)$ is Schur-geometrically convex (Schur-geometrically concave) with respect to $(a,b) \in \mathbb{R}^2_+$ if and only if $p+q \leq (\geq) 0$;

(2) If $q < p$, then $V_{p,q}(a,b)$ is Schur-geometrically convex (Schur-geometrically concave) with respect to $(a,b) \in \mathbb{R}^2_+$ if and only if $p+q \geq (\leq) 0$.

Taking $q = p+1$ in Theorems 3.2 – 3.4, we have the following result.

Corollary 3.5. *Let $m \in \mathbb{R}$ and some fixed $p \in \mathbb{R}$.*

(1) If $m > 0$, then $V_p(a,b)$ is Schur m-power convex (Schur m-power concave) with respect to $(a,b) \in \mathbb{R}^2_+$ if and only if $2p+1 \leq (\geq) -m$ and $p+1 \leq (\geq) 0$;

(2) If $m < 0$, then $V_p(a,b)$ is Schur m-power convex (Schur m-power concave) with respect to $(a,b) \in \mathbb{R}^2_+$ if and only if $2p+1 \leq (\geq) -m$ and $p \leq (\geq) 0$;

(3) If $m = 0$, then $V_p(a,b)$ is Schur-geometrically (Schur-geometrically concave) with respect to $(a,b) \in \mathbb{R}^2_+$ if and only if $2p+1 \leq (\geq) 0$.

Putting $p,q \in \mathbb{N}$ in Theorems 3.2 – 3.4, we have

Corollary 3.6. *For $m \in \mathbb{R}$ and $p,q \in \mathbb{N}$ with $p<q$.*

(1) *If $m>0$, then $V_{p,q}(a,b)$ is Schur m-power concave with $(a,b)\in\mathbb{R}^2_+$;*

(2) *If $m<0$, then $V_{p,q}(a,b)$ is Schur m-power concave with $(a,b)\in\mathbb{R}^2_+$ if and only if $p+q\geq -m$;*

(3) *If $m=0$, then $V_{p,q}(a,b)$ is Schur-geometrically concave $(a,b)\in\mathbb{R}^2_+$.*

Remark 3.7. Clearly, Corollary 3.5 and Corollary 3.6 are generalizations of Theorem 1.3 and Theorem 1.4 respectively.

Next, we will use Theorem 3.2, 3.3 and 3.4 to establish some new inequalities involving generalized invariant contra harmonic means, arithmetic mean and geometric mean.

For $m,p,q\in\mathbb{R}$ and $a,b\in\mathbb{R}_+$. Using the (2.1) and (2.2), we obtain

$$\big(\ln G(a,b),\ln G(a,b)\big) \prec \big(\ln a,\ln b\big),\ \text{ for }\ m\neq 0$$

and

$$\big(A(a^m,b^m),A(a^m,b^m)\big) \prec \big(a^m,b^m\big),\ \text{ for }\ m\neq 0,$$

where $A(a,b)=\frac{a+b}{2}$ and $G(a,b)=\sqrt{ab}$ are the arithmetic mean and geometric mean.

By the generalized invariant contra harmonic means (1.2) and the Theorem 3.4(1), we acquire

Theorem 3.8. *Let $p,q\in\mathbb{R}$ with $p<q$, the inequality*

$$[G(a,b)]^{q-p}(a^p+b^p)\geq(\leq)a^q+b^q \tag{3.2}$$

holds for any $a,b\in\mathbb{R}_+$ if and only if $p+q\leq(\geq)0$ and $p\leq(\geq)0$, where $G(a,b)$ is the geometric mean.

Using the Theorems 3.2(1) and 3.3(1), we have

Theorem 3.9. *Let $m,p,q\in\mathbb{R}$ with $m\neq 0$ and $p<q$.*

1. *If $m>0$, the inequality*

$$[G(a,b)]^{q-p}[a^p+b^p]\geq(\leq)[A(a^m,b^m)]^{\frac{q-p}{m}}[a^q+b^q] \tag{3.3}$$

holds for any $a,b\in\mathbb{R}_+$ if and only if $p+q\leq(\geq)-m$ and $q\leq(\geq)0$;

2. *If $m<0$, the inequality*

$$[G(a,b)]^{q-p}[a^p+b^p]\geq(\leq)[A(a^m,b^m)]^{\frac{q-p}{m}}[a^q+b^q] \tag{3.4}$$

holds for any $a,b\in\mathbb{R}_+$ if and only if $p+q\leq(\geq)-m$ and $p\leq(\geq)0$,

where $A(a,b)$ and $G(a,b)$ are the arithmetic mean and geometric mean.

The identity mean and logarithmic mean of $a,b\in\mathbb{R}_+$ are respectively defined by

$$L(a,b)=\begin{cases}\dfrac{b-a}{\ln b-\ln a}, & a\neq b,\\ a, & a=b.\end{cases}\qquad I(a,b)=\begin{cases}\dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & a\neq b,\\ a, & a=b,\end{cases}$$

It is well known that the following interesting inequalities hold:

$$H(a,b)\leq G(a,b)\leq L(a,b)\leq I(a,b)\leq A(a,b),\quad a,b\in\mathbb{R}_+. \tag{3.5}$$

Based on the inequalities (3.2), (3.3), (3.4) and (3.5), we propose the following open problem.

Open problem 3.1. Let $a,b\in\mathbb{R}_+$ and $p,q\in\mathbb{R}$ with $p<q$. What conditions must p and q satisfy for the inequalities

$$V_{p,q}(a,b)\leq I(a,b)\quad and\quad V_{p,q}(a,b)\leq L(a,b)$$

to hold ?

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Use of AI Tools

AI tools were not employed in generating, analyzing, or interpreting the results.

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