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Research Article

Extending and Improving Monotonic Integral Inequalities

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Abstract

Monotonic integral inequalities dealing with the product of two functions are standard in mathematics. In this article, we contribute to the subject by considering new assumptions. They innovate by taking into account the possible interaction of the involved functions, characterized by inequalities that mix primitive-like operators, derivatives and a certain parameter. This parameter plays a crucial role in the obtained bounds and their refinements.

Keywords: Integral inequalities, primitive, monotonicity, integral decompositions, second mean value theorem for definite integrals

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1. Introduction

Integral inequalities are standard tools of mathematical analysis with applications in probability, statistics, optimization, partial differential equations, physics and engineering. They essentially provide upper and lower bounds on integrals, depending on the properties of the functions involved. Among the most famous integral inequalities are the Cauchy-Schwarz, Hölder, Steffensen, Hermite-Hadamard, Chebychev and (generalized) Minkowski integral inequalities. Each of these gives specific relationships between functions and their integrals. For more details, see [1–6].

Integrals involving products of different types of functions are common in both theoretical and applied contexts. Consequently, they have attracted attention, especially from an inequality point of view. The resulting bounds are typically influenced by the individual behavior of the functions. In particular, the consideration of a monotonic function and an integrable function is an interesting case to study. On this subject, we refer to [7–17].

The focus of this article is to derive new bounds for integrals of the product of two functions of the following form " $\int_a^b p(t)q(t)dt$", where p is a positive, monotonic and differentiable function, and q is an integrable function. Well-known inequalities can be derived using standard sign and monotonicity assumptions on p and q. For example, if q is positive, continuous and monotonic, the following inequality holds:

$$\int_a^b p(t)q(t)dt \geq \min[q(b),q(a)] \int_a^b p(t)dt. \tag{1.1}$$

This is what we call a "monotonic inequality". In full generality, the bounds derived from monotonic inequalities are often restrictive, relying on assumptions such as the positivity or the monotonicity of q. In this article, we consider

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such a classical framework, but beyond these standard assumptions on q. We study refined integral inequalities under new assumptions that relate the functions p and q. These assumptions involve primitive-like operators on q, the derivative of p, and a certain parameter. This interaction is the key of our study. To fix the idea, the following statement will be one of the assumptions: "We assume that there exists $\kappa \in \mathbb{R}$ such that, for any $x \in [a,b]$, we have

$$\int_x^b q(t)dt \geq \frac{\kappa}{p'(x)} p(x)q(x)."$$

The parameter κ will have a direct effect on the bounds obtained, which may become sharper than those in Equation (1.1). We thus improve our understanding of the interaction between p and q in a classical integral inequality framework, but with innovative assumptions.

The rest of the article is as follows: Section 2 presents the problem in more detail, the integral results and a discussion of their consequences. Conclusions and future directions are given in Section 3.

2. Integral results

2.1. Problem

The problem of the study is described below. Let $a,b \in \mathbb{R}$ with $a < b$, p be a positive differentiable monotonic function defined on $[a,b]$, and q be an integrable function defined on $[a,b]$. To fix the idea, we assume that p is non-decreasing and q is positive. Then the following monotonic integral inequality holds:

$$\int_a^b p(t)q(t)dt \geq p(a) \int_a^b q(t)dt. \tag{2.1}$$

This follows from the non-decreasing property of p: the smallest value of p on $[a,b]$ is $p(a)$, i.e., $p(x) \geq \inf_{t\in[a,b]} p(t) = p(a)$ for any $x \in [a,b]$, and the positivity of q which preserves this inequality. Without further information about p and q, the lower bound obtained is considered to be sharp. However, with the ideas of extension and improvement in mind, the following questions naturally arise:

1. Can we prove a similar result by relaxing the positivity assumption on q?
   In particular, can we establish a lower bound for $\int_a^b p(t)q(t)dt$ when q can take negative values?
2. Is it possible to improve the current result?
   In particular, can we derive a better lower bound that is proportional to $p(a)\int_a^b q(t)dt$, but with an improved proportionality factor that reflects the properties of p and q?

These questions implicitly ask how the individual and possible relationships of p and q can affect a well-known monotonic inequality. Indeed, removing the positivity constraint on q would increase the generality and applicability of the result. Furthermore, a more refined lower bound is always welcome in applications. The rest of this section provides possible answers to these questions. It also considers lower and upper bounds, thus completing the problem to a more general inequality framework.

2.2. Two theorems

Under the assumption that p is non-decreasing, the result below gives refined bounds for $\int_a^b p(t)q(t)dt$. They are derived under a technical assumption that relates the integral of q over $[x,b]$ to the values of p, p' and q at a point x.

Theorem 2.1. *Let $a,b \in \mathbb{R}$ with $a < b$, p be a positive differentiable non-decreasing function defined on $[a,b]$ and q be an integrable function defined on $[a,b]$. We assume that there exists $\kappa \in \mathbb{R}$ such that, for any $x \in [a,b]$, we have*

$$\int_x^b q(t)dt \geq \frac{\kappa}{p'(x)} p(x)q(x). \tag{2.2}$$

1. *If $\kappa < 1$, then the following inequality holds:*

$$\int_a^b p(t)q(t)dt \geq \frac{p(a)}{1-\kappa} \int_a^b q(t)dt. \tag{2.3}$$

2. *If $\kappa > 1$, then the following inequality holds:*

$$\int_a^b p(t)q(t)dt \leq \frac{p(a)}{1-\kappa}\int_a^b q(t)dt. \tag{2.4}$$

3. *If $\kappa = 1$, then we have*

$$p(a)\int_a^b q(t)dt \leq 0. \tag{2.5}$$

Furthermore, if the inequality in Equation (2.2) is reversed, then the inequalities in Equations (2.3), (2.4) and (2.5) are also reversed.

Proof. The results in the three items have the same mathematical foundation. Since p is differentiable, for any $t \in [a,b]$, we can express p as follows:

$$p(t) = [p(t) - p(a)] + p(a) = \int_a^t p'(u)du + p(a).$$

Based on this expression and a change in the order of integration, made possible by the Fubini-Tonelli integral theorem, we get

$$\begin{aligned}\int_a^b p(t)q(t)dt &= \int_a^b \left[\int_a^t p'(u)du + p(a)\right] q(t)dt \\ &= \int_a^b \int_a^t p'(u)q(t)dudt + p(a)\int_a^b q(t)dt \\ &= \int_a^b \int_u^b p'(u)q(t)dtdu + p(a)\int_a^b q(t)dt \\ &= \int_a^b p'(u)\left[\int_u^b q(t)dt\right] du + p(a)\int_a^b q(t)dt. \end{aligned} \tag{2.6}$$

Using the assumption in Equation (2.2) and the fact p is positive, differentiable and non-decreasing, which means that $p'(u) \geq 0$ for any $u \in [a,b]$, we find that

$$\begin{aligned}&\int_a^b p'(u)\left[\int_u^b q(t)dt\right] du + p(a)\int_a^b q(t)dt \\ &\geq \int_a^b p'(u)\frac{\kappa}{p'(u)}p(u)q(u)du + p(a)\int_a^b q(t)dt \\ &= \kappa\int_a^b p(u)q(u)du + p(a)\int_a^b q(t)dt. \end{aligned} \tag{2.7}$$

The key to the proof is the finding of the main integral term $\int_a^b p(u)q(u)du$ multiplied by κ into the lower bound. Combining Equations (2.6) and (2.7), and applying a standardization of the notation, we get

$$\int_a^b p(t)q(t)dt \geq \kappa\int_a^b p(t)q(t)dt + p(a)\int_a^b q(t)dt,$$

so that

$$(1-\kappa)\int_a^b p(t)q(t)dt \geq p(a)\int_a^b q(t)dt.$$

Thanks to this inequality, the conclusions below hold.

1. If $\kappa < 1$, dividing by $1-\kappa > 0$, we find that
$$\int_a^b p(t)q(t)dt \geq \frac{p(a)}{1-\kappa}\int_a^b q(t)dt.$$
2. If $\kappa > 1$, dividing by $1-\kappa < 0$, we get
$$\int_a^b p(t)q(t)dt \leq \frac{p(a)}{1-\kappa}\int_a^b q(t)dt.$$
3. Clearly, if $\kappa = 1$, then $1-\kappa = 0$, which implies that
$$p(a)\int_a^b q(t)dt \leq 0.$$

If the inequality in Equation (2.2) is reversed, the inequality in Equation (2.7) is reversed, implying that the final inequalities are also reversed. The desired results are also established. □

In a sense, the theorem below completes Theorem 2.1. Under the assumption that p is non-increasing, it establishes refined bounds for $\int_a^b p(t)q(t)dt$. They are derived under a technical assumption that relates the integral of q over $[a,x]$ to the values of p, p' and q at a point x.

Theorem 2.2. *Let $a,b \in \mathbb{R}$ with $a < b$, p be a positive differentiable non-increasing function defined on $[a,b]$ and q be an integrable function defined on $[a,b]$. We assume that there exists $\tau \in \mathbb{R}$ such that, for any $x \in [a,b]$, we have*
$$\int_a^x q(t)dt \geq -\frac{\tau}{p'(x)}p(x)q(x). \tag{2.8}$$

1. *If $\tau < 1$, then the following inequality holds:*
$$\int_a^b p(t)q(t)dt \geq \frac{p(b)}{1-\tau}\int_a^b q(t)dt. \tag{2.9}$$
2. *If $\tau > 1$, then the following inequality holds:*
$$\int_a^b p(t)q(t)dt \leq \frac{p(b)}{1-\tau}\int_a^b q(t)dt. \tag{2.10}$$
3. *If $\tau = 1$, then we have*
$$p(b)\int_a^b q(t)dt \leq 0. \tag{2.11}$$

Furthermore, if the inequality in Equation (2.8) is reversed, then the inequalities in Equations (2.9), (2.10) and (2.11) are also reversed.

Proof. The results in the three items have the same mathematical foundation. Since p is differentiable, for any $t \in [a,b]$, we can write
$$p(t) = p(b) - [p(b) - p(t)] = p(b) - \int_t^b p'(u)du.$$
With this and a change in the order of integration, which is possible thanks to the Fubini-Tonelli integral theorem, we have
$$\int_a^b p(t)q(t)dt = \int_a^b \left[p(b) - \int_t^b p'(u)du\right] q(t)dt$$
$$= p(b)\int_a^b q(t)dt - \int_a^b \int_t^b p'(u)q(t)dudt$$

$$= p(b)\int_a^b q(t)dt - \int_a^b \int_a^u p'(u)q(t)dtdu$$

$$= p(b)\int_a^b q(t)dt + \int_a^b [-p'(u)]\left[\int_a^u q(t)dt\right]du. \tag{2.12}$$

Using Equation (2.8) and the fact that p is differentiable and non-increasing, which means that $p'(u) \leq 0$, so $-p'(u) \geq 0$, for any $u \in [a,b]$, we get

$$p(b)\int_a^b q(t)dt + \int_a^b [-p'(u)]\left[\int_a^u q(t)dt\right]du$$

$$\geq p(b)\int_a^b q(t)dt + \int_a^b [-p'(u)]\left[-\frac{\tau}{p'(u)}p(u)q(u)\right]du$$

$$= p(b)\int_a^b q(t)dt + \tau\int_a^b p(u)q(u)du. \tag{2.13}$$

The presence of the main integral term $\int_a^b p(u)q(u)du$ multiplied by τ into the lower bound is central for the rest of the proof. It follows from Equations (2.12) and (2.13), and a standardization of the notation that

$$\int_a^b p(t)q(t)dt \geq p(b)\int_a^b q(t)dt + \tau\int_a^b p(t)q(t)dt,$$

so that

$$(1-\tau)\int_a^b p(t)q(t)dt \geq p(b)\int_a^b q(t)dt.$$

Thanks to this inequality, the conclusions below hold.

1. If $\tau < 1$, dividing by $1-\tau > 0$, we get
$$\int_a^b p(t)q(t)dt \geq \frac{p(b)}{1-\tau}\int_a^b q(t)dt.$$
2. If $\tau > 1$, dividing by $1-\tau < 0$, we obtain
$$\int_a^b p(t)q(t)dt \leq \frac{p(b)}{1-\tau}\int_a^b q(t)dt.$$
3. Clearly, if $\tau = 1$, since $1-\tau = 0$, we have
$$p(b)\int_a^b q(t)dt \leq 0.$$

If the inequality in Equation (2.8) is reversed, the inequality in Equation (2.13) is reversed, implying that the final inequalities are reversed. The desired results are established. □

We can compare Theorem 2.2 with what can be obtained by the second mean value theorem for definite integrals. A statement of this classical theorem is given below. Let $a,b \in \mathbb{R}$ with $a < b$, p be a positive differentiable non-increasing function defined on $[a,b]$ and q be an integrable function defined on $[a,b]$. Then there exists $c \in (a,b)$ such that

$$\int_a^b p(t)q(t)dt = p(a)\int_a^c q(t)dt.$$

If we consider the assumption in Equation (2.8), the following inequality holds:

$$\int_a^b p(t)q(t)dt \geq -p(a)\frac{\tau}{p'(c)}p(c)q(c).$$

The two results obtained are different from those of Theorem 2.2, which deals with $\int_a^b q(t)dt$ in the lower bound, without an undefined value like c. We thus provide alternative solutions to those derived from some standard approaches.

Let us now discuss more deeply the interest of our theorems in the subsection below.

2.3. Discussion

The originality in Theorems 2.1 and 2.2 lies mainly in the statements of the assumptions in Equations (2.2) and (2.8), and the bounds obtained. First, it must be noted that these assumptions do not imply a constant sign for q. So we are in a more complex setting than the standard monotonicity inequalities. Second, the factor constant κ captures the extent to which the local behavior of p, p' and q determines the contribution of q to the main integral. As κ tends to 1, the proportionality constant $1/(1-\kappa)$ becomes larger, which has a direct effect on the obtained bounds. Third, as an example of improvement, applying Theorem 2.1 with q positive and assuming that $\kappa \in [0,1)$, we clearly have

$$\int_a^b p(t)q(t)dt \geq \frac{p(a)}{1-\kappa}\int_a^b q(t)dt \geq p(a)\int_a^b q(t)dt.$$

Thus, our results improve on the analogous monotonic inequality as recalled in Equation (2.1), under new assumptions on p and q. Many more examples can be given based on Theorems 2.1 and 2.2.

3. Conclusion and future directions

The problem of extending and improving basic monotonicity integral inequalities has been addressed in this article. Two theorems have been established, i.e., Theorems 2.1 and 2.2, under different assumptions, i.e., described in Equations (2.2) and (2.8), which have the originality of taking into account a possible interaction of the functions involved. Several bounds have been established and discussed, highlighting the contributions to the subject.

For future work, one can think of generalizing our assumptions. In particular, an open question is whether the dependence on p' can be relaxed, or whether analogous results hold for non-monotonic functions p. In addition, exploring the implications of our inequalities in applied contexts, such as weighted integrals in probability theory or physics, could provide valuable insights. We leave these lines of research for future investigation.

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