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Research Article

Convergence of the Split Common Fixed Point Problem for Multi-Output Sets with k-Semi-Contractive Mappings

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Abstract

This paper introduces a novel iterative scheme that integrates an inertial technique and Nesterov acceleration for solving the multi-output set split common fixed point problem with k-semi-contractive mappings. The primary objectives are to enhance the convergence rate and reduce computational complexity. Firstly, under the condition of fixed step size, two iterative algorithms are designed, and their weak convergence and strong convergence are proved respectively. The strong convergence result is achieved by introducing relaxation parameters and external control sequences. Secondly, to avoid the difficulty of operator norm calculation, a variable step size iterative scheme is further proposed, which improves the practicality of the algorithm by dynamically adjusting the step size, and its convergence is strictly analyzed. The theoretical proof shows that under the conditions of satisfying the subclosed principle and appropriate parameters, the sequence generated by the proposed algorithm can converge to the solution of the problem. Numerical experiments verify the acceleration effect of the inertial term and the advantages of variable step size.

Keywords: Multi-output set split common fixed point problems, k-semi-contractive mapping, inertial algorithm, Nesterov acceleration, variable step size.

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1. Introduction

Let $\mathscr{H}$ and $\mathscr{H}_1$ be two Hilbert spaces, and $A:\mathscr{H}\to\mathscr{H}_1$ be a bounded linear operator. Let $T_0:\mathscr{H}\to\mathscr{H}$ and $T_1:\mathscr{H}_1\to\mathscr{H}_1$ be two nonlinear operators. The split common fixed point problem [1] aims to find $x^*\in\mathscr{H}$ such that

$$x^*\in F(T_0)\quad\text{and}\quad Ax^*\in F(T_1). \tag{1.1}$$

Where $F(T_0)$ denotes the fixed point set of T_0, and $A^{-1}(F(T_1))=\{x\in\mathscr{H}:Ax\in F(T_1)\}$.

In recent decades, the split common fixed point problem has been generalized in various ways. A significant extension is the multi-set split common fixed point problem [2], defined as follows: For each $i=1,2,\dots N$, let $\mathscr{H}_i$ be a Hilbert space, $A_i:\mathscr{H}\to\mathscr{H}_i$ be bounded linear operators, and $T_i:\mathscr{H}_i\to\mathscr{H}_i$ be nonlinear operators. The problem aims to find $x^*\in\mathscr{H}$ such that

$$x^*\in F(T_0)\bigcap[\bigcap_{i=1}^{N}A_i^{-1}(F(T_i))]. \tag{1.2}$$

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Where $F(T_i)$ denotes the fixed point set of T_i, and $A_i^{-1}(F(T_i)) = \{x \in \mathscr{H} : A_i x \in F(T_i)\}$ for each $i = 1,2,\ldots N$. When $N = 1$, the multi-output set split common fixed point problem reduces to the split common fixed point problem. Throughout this paper, we assume that the solution set, denoted by S, of the multi-output set split common fixed point problem is nonempty.

In 2023, Cui and Wang [3] addressed the above problem for the case when nonlinear operators are semi-contractive operators. To find its solution, they proposed the following iterative method:

$$x_{n+1} = x_n - \tau \sum_{i=0}^{N} A_i^*(A_i x_n - T_i(A_i x_n)). \tag{1.3}$$

Where τ is an appropriate step size, A_i^* is the conjugate operator of A_i, and they proved its weak convergence under the condition $0 < \tau(\sum_{i=0}^{N} \|A_i\|^2) < k$, as well as

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\left[x_n - \tau \sum_{i=0}^{N} A_i^*(A_i x_n - T_i(A_i x_n))\right]. \tag{1.4}$$

Where $\{\alpha_n\} \subseteq (0,1]$, and proved its strong convergence. However, these methods require the calculation of operator norms, which is difficult in practice. Subsequently, they proposed a variable step size iterative method by replacing τ with τ_n, setting the variable step size as:

$$\tau_n = \frac{k \sum_{i=0}^{N} \|A_i x_n - T_i(A_i x_n)\|^2}{2\|\sum_{i=0}^{N} A_i^*(I-T_i)A_i x_n\|^2}$$

and proved the weak and strong convergence of the above iterative methods respectively.

Inertial algorithms serve as a significant acceleration technique widely applied in iterative solutions for optimization problems and fixed-point problems. The core idea of inertial technology is to incorporate information from the previous iteration point into the current one, specifically by generating a momentum term through the difference between the current and previous points, thereby accelerating the convergence rate of the algorithm. The general form of the inertial term is expressed as:

$$y_n = x_n + \beta_n(x_n - x_{n-1})$$

where β_n is the inertial parameter. This technique effectively addresses the slow convergence of traditional gradient-based methods, demonstrating remarkable advantages, particularly when dealing with large-scale or ill-conditioned problems.

Based on the above work, this paper further improves the iterative method for the multi-output set split common fixed point problem under semi-contractive operators. By combining inertial algorithm with Nesterov acceleration method, we accelerate the algorithm speed and prove its weak and strong convergence. Numerical experiments verify that the acceleration effect of the inertial term and the advantages of variable step size are very significant.

2. Preliminaries

Definition 2.1. [4] Let $k < 1$. If for any $x, y \in \mathscr{H}$,

$$\|Tx - Ty\|^2 \le \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2$$

. Where I is the identity mapping, then when $k = -1$, the operator T is called a firmly non-expansive mapping; when $k = 0$, it is called a non-expansive mapping; when $k \in (0,1)$, it is called a k-strictly pseudo-contractive mapping.

Definition 2.2. [5] Let $F(T) \neq \varnothing$ and $k < 1$. If for any $(x,z) \in \mathscr{H} \times F(T)$

$$\|Tx - z\|^2 \le \|x-z\|^2 + k\|x - Tx\|^2,$$

then when $k = -1$, T is called a firmly quasi-non-expansive mapping; when $k = 0$, it is called a quasi-non-expansive mapping; when $k \in (0,1)$, it is called a k-semi-contractive mapping.

Remark 2.1. Let T be a k-semi-contractive mapping,then $F(T)$ is a closed convex subset from [6].

Definition 2.3. [7] We say that T satisfies the demi-closed principle if $I-T$ is demi-closed at 0, i.e., for any sequence $\{x_n\} \subseteq \mathscr{H}$ and $x^* \in \mathscr{H}$, we have:

$$x_n \rightharpoonup x^* \quad \text{and} \quad (I-T)x_n \to 0 \quad \Rightarrow \quad x^* \in F(T).$$

Where $\rightharpoonup$ denotes weak convergence and $\to$ denotes strong convergence.

Lemma 2.1. *[8] Let $x \in \mathscr{H}$, then $y = P_C x$ if and only if $y \in C$ and satisfies $\langle x-y, z-y \rangle \leq 0, \forall z \in C$.*

Lemma 2.2. *[9] Let $\{a_n\}$ be a nonnegative real sequence satisfying*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n, n \geq 0.$$

Where $\{\alpha_n\} \subset (0,1)$, $\{\delta_n\}$ is a real sequence. If $\sum_{n=0}^{\infty} \alpha_n = +\infty$ and $\overline{\lim}_{n\to\infty} \delta_n \leq 0$, then $\lim_{n\to+\infty} a_n = 0$.

Lemma 2.3. *[3] Let $Q = \{0,1,2,\ldots,N\}$, $k = \min_{i\in Q}(1-k_i)$, and $T = I - \tau\sum_{i=0}^{N} A_i^*(A_i - T_i(A_i))$, where A_0 is the identity mapping on $\mathscr{H}$. If T_i is a k_i-semi-contractive mapping $(i \in Q)$, then for any $(x,z) \in \mathscr{H} \times S$, the following inequality holds:*

$$\|Tx - z\|^2 \leq \|x-z\|^2 - \tau\left(k - \tau\sum_{i=0}^{N}\|A_i\|^2\right)\sum_{i=0}^{N}\|A_i x - T_i(A_i x)\|^2.$$

Moreover, if $0 < \tau(\sum_{i=0}^{N}\|A_i\|^2) < k$, then $S = F(T)$.

3. Weak and strong convergence of iterative algorithms with fixed step size

Lemma 3.1. *Let the nonnegative sequence $\{\alpha_n\}$ satisfy $\alpha_{n+1} \leq (1+\beta_n)\alpha_n + \beta_n\alpha_{n-1} (n = 1,2,\ldots)$, where the nonnegative sequence $\{\beta_n\}$ satisfies $\sum_{n=1}^{\infty}\beta_n < +\infty$, then the sequence $\{\alpha_n\}$ is bounded.*

Proof. Let $M_n = \max\{\alpha_0, \alpha_1, \ldots \alpha_n\}$, then we have $0 \leq \alpha_n \leq M_n$ and $0 \leq \alpha_{n-1} \leq M_n$, which gives:

$$\alpha_{n+1} \leq (1+\beta_n)M_n + \beta_n M_n = (1+2\beta_n)M_n.$$

Note that

$$M_{n+1} = \max\{\alpha_0, \alpha_1, \ldots, \alpha_n, \alpha_{n+1}\} = \max\{M_n, \alpha_{n+1}\},$$

and

$$0 \leq \alpha_i \leq M_n (i = 0,1,\ldots,n), \quad \alpha_{n+1} \leq (1+2\beta_n)M_n.$$

Therefore, $M_{n+1} \leq (1+2\beta_n)M_n$.

Using the inequality $1+x \leq e^x (\forall x \geq 0)$, we have:

$$\begin{aligned} M_n &\leq (1+2\beta_{n-1})(1+2\beta_{n-2})\cdots(1+2\beta_1)M_1 \\ &\leq e^{2\beta_{n-1}} \cdot e^{2\beta_{n-2}} \cdot \cdots \cdot e^{2\beta_1} M_1 \\ &= e^{2(\beta_{n-1}+\beta_{n-2}+\cdots+\beta_1)} M_1. \end{aligned}$$

Since $\sum_{n=1}^{\infty}\beta_n < +\infty$, let $\sum_{n=0}^{\infty}\beta_n = D$, then $\sum_{k=1}^{n-1} 2\beta_k \leq 2D$, therefore:

$$0 \leq M_n \leq e^{2D} M_1 = e^{2D}\max\{\alpha_0, \alpha_1\}.$$

Hence, for all $n = 1,2,\ldots$, there exists a real number M such that $M_n \leq M$. This completes the proof that the sequence $\{\alpha_n\}$ is bounded.

For convenience, we set $Q = \{0,1,2,\ldots,N\}$, $k = \min_{i\in Q}(1-k_i)$, and assume that $k_i \in (0,1), \forall i \in Q$.

Algorithm 3.1. Choose an arbitrary initial point $x_0 \in \mathscr{H}$, and set $x_1 = x_0$. Given the current iteration x_n, compute the next iteration x_{n+1} as follows:

$$\begin{cases} y_n = x_n + \beta_n(x_n - x_{n-1}), \\ z_n = y_n - \tau \sum_{i=0}^{N} A_i^* \left(A_i y_n - T_i(A_i y_n)\right), \\ x_{n+1} = z_n. \end{cases}$$

where τ is an appropriately chosen step size.

Theorem 3.1. *Let T_i be k_i-semi-contractive mappings $(\forall i \in Q)$ satisfying the demi-closed principle, $0 < \tau(\sum_{i=0}^{N} \|A_i\|^2) < k$, and $\sum_{n=0}^{\infty} \beta_n < +\infty$. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 weakly converges to an element in the solution set S of problem* (1.2).

Proof. First, we prove that $\{x_n\}$ and $\{y_n\}$ are bounded sequences.

By Lemma 2.3 and Algorithm 3.1, we have:

$$\|x_{n+1} - z\|^2 = \|z_n - z\|^2 \le \|y_n - z\|^2 - \tau \left(k - \tau \sum_{i=0}^{N} \|A_i\|^2\right) \sum_{i=0}^{N} \|A_i y_n - T_i(A_i y_n)\|^2. \tag{3.1}$$

Since $0 < \tau(\sum_{i=0}^{N} \|A_i\|^2) < k$, we have $\|x_{n+1} - z\|^2 \le \|y_n - z\|^2$, i.e., $\|x_{n+1} - z\| \le \|y_n - z\|$. Since $y_n = x_n + \beta_n(x_n - x_{n-1})$, we get:

$$\begin{aligned} \|y_n - z\| &= \|x_n - z + \beta_n (x_n - x_{n-1})\| \\ &\le \|x_n - z\| + \beta_n \|x_n - z - (x_{n-1} - z)\| \\ &\le \|x_n - z\| + \beta_n (\|x_n - z\| + \|x_{n-1} - z\|) \\ &= (1 + \beta_n) \|x_n - z\| + \beta_n \|x_{n-1} - z\|. \end{aligned} \tag{3.2}$$

Thus, from $\|x_{n+1} - z\| \le \|y_n - z\|$ and the above inequality, we obtain:

$$\|x_{n+1} - z\| \le (1+\beta_n) \|x_n - z\| + \beta_n \|x_{n-1} - z\|. \tag{3.3}$$

Since $\sum_{n=0}^{\infty} \beta_n < \infty$, using inequality (3.3) and Lemma 3.1, we find that $\{\|x_n - z\|\}$ is bounded. Therefore, $\{x_n\}$ is bounded. From (3.2), we see that $\{y_n\}$, $\{A_i y_n\}$, $\{T_i(A_i y_n)\}$ are bounded. Hence, there exists $G > 0$ such that:

$$0 \le \max\{\|x_n - z\|, \|x_n\|, \|y_n\|, \|A_i x_n\|, \|T_i(A_i x_n)\|\} \le G, \quad (n = 0, 1, 2 \ldots, i = 0, 1, 2 \ldots).$$

Next, we prove that $A_i x_n - T_i(A_i x_n) \to 0 (n \to \infty, i = 0, 1, 2 \ldots N)$. Since

$$\begin{aligned} \|y_n - z\| &\le \|x_n - z\| + \beta_n \|x_n - x_{n-1}\| \\ &\le \|x_n - z\| + 2G\beta_n, \end{aligned}$$

We have

$$\begin{aligned} \|y_n - z\|^2 &\le (\|x_n - z\| + 2G\beta_n)^2 \\ &= \|x_n - z\|^2 + 4G^2\beta_n^2 + 4G\beta_n \|x_n - z\| \\ &\le \|x_n - z\|^2 + 4G^2\beta_n^2 + 4G^2\beta_n \\ &= \|x_n - z\|^2 + 4G^2\beta_n (1 + \beta_n). \end{aligned}$$

Since $\sum_{n=0}^{\infty} \beta_n < +\infty$, we have $\lim_{n\to\infty} \beta_n = 0$. Therefore, there exists $n_0 \in \mathbb{N}^+$ such that when $n \ge n_0$, we have $0 \le \beta_n < 1$, so:

$$\|y_n - z\|^2 \le \|x_n - z\|^2 + 8G^2\beta_n \quad (n = n_0, n_0 + 1, \ldots),$$

Combining with (3.1), we have:

$$\|x_{n+1}-z\|^2 \leq \|x_n-z\|^2 + 8G^2\beta_n - \tau\left(k-\tau\sum_{i=0}^{N}\|A_i\|^2\right)\sum_{i=0}^{N}\|A_iy_n - T_i(A_iy_n)\|^2.$$

Thus

$$\tau\left(k-\tau\sum_{i=0}^{N}\|A_i\|^2\right)\sum_{i=0}^{N}\|A_iy_n - T_i(A_iy_n)\|^2 \leq \|x_n-z\|^2 - \|x_{n+1}-z\|^2 + 8G^2\beta_n, (n=n_0,n_0+1,...).$$

Then

$$\begin{aligned}
&\sum_{n=n_0}^{\infty}\tau\left(k-\tau\sum_{i=0}^{N}\|A_i\|^2\right)\sum_{i=0}^{N}\|A_iy_n - T_i(A_iy_n)\|^2 \\
&\leq \sum_{n=n_0}^{\infty}\left(\|x_n-z\|^2 - \|x_{n+1}-z\|^2\right) + \sum_{n=n_0}^{\infty}8G^2\beta_n \\
&= \|x_{n_0}-z\|^2 - \lim_{n\to\infty}\|x_{n+1}-z\|^2 + \sum_{n=0}^{\infty}8G\beta_n \\
&\leq \|x_{n_0}-z\|^2 + 8GS < +\infty.
\end{aligned}$$

Note that $\tau(k-\tau\sum_{i=0}^{N}\|A_i\|^2) > 0$, so the series $\tau(k-\tau\sum_{i=0}^{N}\|A_i\|^2)\sum_{n=n_0}^{\infty}(\sum_{i=0}^{N}\|A_iy_n - T_i(A_iy_n)\|^2)$ converges, hence:

$$\lim_{n\to\infty}\sum_{i=0}^{N}\|A_iy_n - T_i(A_iy_n)\|^2 = 0.$$

Therefore:

$$A_iy_n = T_i(A_iy_n), \quad \forall i \in Q.$$

Since $\|y_n - x_n\| = \beta_n\|x_n - x_{n-1}\| \to 0 (n\to\infty)$, we have $y_n - x_n \to 0 (n\to\infty)$. Note that A_iy_n is the fixed point of operator T_i, therefore:

$$\begin{aligned}
\|A_ix_n - T_i(A_ix_n)\|^2 &= \|A_i(x_n-y_n) + A_iy_n - T_i(A_ix_n)\|^2 \\
&= \|A_i(x_n-y_n) + T_i(A_iy_n) - T_i(A_ix_n)\|^2 \\
&\leq (\|A_i(x_n-y_n)\| + \|T_i(A_iy_n) - T_i(A_ix_n)\|)^2 \\
&= \|A_i(x_n-y_n)\|^2 + 2\|A_i(x_n-y_n)\|\,\|T_i(A_iy_n) - T_i(A_ix_n)\| \\
&\quad + \|T_i(A_iy_n) - T_i(A_ix_n)\|^2 \\
&= \|A_i(x_n-y_n)\|^2 + 2\|A_i(x_n-y_n)\|\,\|A_iy_n - T_i(A_ix_n)\| \\
&\quad + \|A_iy_n - T_i(A_ix_n)\|^2.
\end{aligned}$$

Since T_i is a k_i-semi-contractive mapping, we have:

$$\begin{aligned}
\|A_iy_n - T_i(A_ix_n)\|^2 &\leq \|A_iy_n - A_ix_n\|^2 + k_i\|(I-T_i)A_ix_n\|^2 \\
&= \|A_i(y_n-x_n)\|^2 + k_i\|A_ix_n - T_i(A_ix_n)\|^2.
\end{aligned}$$

Then

$$\|A_ix_n - T_i(A_ix_n)\|^2 \leq \|A_iy_n - A_ix_n\|^2 + 2\|A_i(y_n-x_n)\|\,\|A_iy_n - T_i(A_ix_n)\|$$

$$+\|A_iy_n - A_ix_n\|^2 + k_i\|A_ix_n - T_i(A_ix_n)\|^2$$
$$\leq 2\|A_i(y_n - x_n)\|^2 + 2\|A_i(y_n - x_n)\|\|A_iy_n - T_i(A_ix_n)\|$$
$$\leq 2\|A_i\|^2\|y_n - x_n\|^2 + 4G\|A_i\|\|y_n - x_n\| \to 0, \quad (n\to\infty).$$

Therefore

$$A_ix_n - T_i(A_ix_n) \to 0, \quad (n\to\infty).$$

Finally, we prove that every weak cluster point of $\{x_n\}$ belongs to S.

Let x^* be any weak cluster point of $\{x_n\}$, then there exists $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup x^*(k\to+\infty)$. Since A_i is a bounded linear operator, we have $A_ix_{n_k} \rightharpoonup A_ix^*$. Also, since $A_ix_{n_k} \to T_i(A_ix_{n_k})$, by the demi-closed principle, we have:

$$A_ix^* = T_i(A_ix^*) \quad \Rightarrow \quad x^* \in A_i^{-1}(F(T_i)).$$

That is, $x^* \in S$. By the arbitrariness of x^*, we conclude that $\{x_n\}$ weakly converges to an element in S.

Lemma 3.2. *Let the nonnegative sequence $\{A_n\}$ satisfy*

$$A_{n+1} \leq [1-\alpha_n(1-\beta_n+\alpha_n\beta_n)]A_n + (1-\alpha_n)\alpha_n\beta_nA_{n-1} + \alpha_nc,$$

where $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = +\infty$, $\lim_{n\to\infty}\beta_n = 0$, and $c > 0$. Then the sequence $\{A_n\}$ is bounded.

Proof. Let $M_n = \max\{A_0, A_1, \ldots, A_n\}$, so we have $0 \leq A_n \leq M_n$ and $0 \leq A_{n-1} \leq M_n$. Then we obtain:

$$\begin{aligned} A_{n+1} &\leq [1-\alpha_n(1-\beta_n+\alpha_n\beta_n)]M_n + (1-\alpha_n)\alpha_n\beta_nM_n + \alpha_nc \\ &= \left(1-\alpha_n+\alpha_n\beta_n-\alpha_n^2\beta_n+\alpha_n\beta_n-\alpha_n^2\beta_n\right)M_n + \alpha_nc \\ &= \left(1-\alpha_n+2\alpha_n\beta_n-2\alpha_n^2\beta_n\right)M_n + \alpha_nc \\ &= (1-\alpha_n)M_n + \alpha_n[2\beta_n(1-\alpha_n)M_n + c] \\ &\leq (1-\alpha_n)M_n + \alpha_n[2|\beta_n|M_n + c]. \end{aligned}$$

Since $\lim_{n\to\infty}\beta_n = 0$, for any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}^+$ such that for all $n \geq N_1$, we have $|\beta_n| < \frac{\varepsilon}{2}$. Let $M = \max\left\{A_0, A_1, \ldots, A_{N_1}, \frac{c}{\varepsilon}\right\}$. We will prove by mathematical induction that $0 \leq A_n \leq M$ for all $n \geq 0$.

Indeed, when $n = 0, 1, \ldots, N_1$, it is obvious that $A_n \leq M$.

Assume that for $n = k$ ($k \geq N_1$), we have $A_i \leq M$ ($i = 0, 1, \ldots, k$), which implies $M_k \leq M$. Then for $n = k+1$, we have:

$$\begin{aligned} A_{k+1} &\leq (1-\alpha_k)M_k + \alpha_k(2|\beta_k|M_k + c) \\ &\leq (1-\alpha_k)M + \alpha_k(\varepsilon M + c). \end{aligned}$$

Since $M \geq \frac{c}{\varepsilon}$, i.e., $c \leq \varepsilon M$, we have:

$$\begin{aligned} A_{k+1} &\leq (1-\alpha_k)M + \alpha_k(\varepsilon M + \varepsilon M) \\ &= (1-\alpha_k+2\alpha_k\varepsilon)M \\ &= M[1-\alpha_k(1-2\varepsilon)] \leq M. \end{aligned}$$

Therefore, the sequence $\{A_n\}$ is bounded. This completes the proof.

Algorithm 3.2. Choose an arbitrary initial point $x_0 \in \mathscr{H}$ and $u \in \mathscr{H}$, and set $x_1 = x_0$. Given the current iteration x_n, compute the next iteration x_{n+1} as follows:

$$\begin{cases} y_n &= x_n + \theta_n(x_n - x_{n-1}), \\ z_n &= y_n - \tau \sum_{i=0}^{N} A_i^* \big(A_i y_n - T_i(A_i y_n)\big), \\ x_{n+1} &= \alpha_n u + (1-\alpha_n) z_n. \end{cases}$$

Where τ is an appropriately chosen step size.

Theorem 3.2. *Let T_i be k_i-semi-contractive mappings $(\forall i \in Q)$ satisfying the demi-closed principle. If the step size τ satisfies $0 < \tau(\sum_{i=0}^{N} \|A_i\|^2) < k$, $\{\alpha_n\} \subset (0,1)$ with $\alpha_n \to 0 (n \to \infty)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\frac{\theta_n}{\alpha_n} \to 0 (n \to +\infty)$, and $\sum_{n=0}^{\infty} \theta_n < +\infty$, then $x_n \to P_S(u) (n \to +\infty)$.*

Proof. Since $z_n = y_n - \tau \sum_{i=0}^{N} A_i^*(A_i y_n - T_i(A_i y_n))$, by Lemma 2.3, we have:

$$\|z_n - c\|^2 \le \|y_n - c\|^2 - \tau \left(k - \tau \sum_{i=0}^{N} \|A_i\|^2 \right) \sum_{i=0}^{N} \|A_i y_n - T_i(A_i y_n)\|^2. \tag{3.4}$$

Thus, $\|z_n - c\| \le \|y_n - c\|, \forall c \in S$. Since $y_n = x_n + \theta_n(x_n - x_{n-1})$, let $a_n = \|x_n - P_S(u)\|^2$. Then:

$$\begin{aligned} \|y_n - P_s(u)\|^2 &= \|x_n - P_s(u) + \theta_n(x_n - x_{n-1})\|^2 \\ &= \|x_n - P_s(u)\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 + 2\theta_n \langle x_n - P_s(u), x_n - x_{n-1} \rangle \\ &\le \|x_n - P_s(u)\|^2 + \theta_n^2 (\|x_n - P_s(u)\| + \|x_{n-1} - P_s(u)\|)^2 \\ &\quad + 2\theta_n \|x_n - P_s(u)\| (\|x_n - P_s(u)\| + \|x_{n-1} - P_s(u)\|) \\ &\le \|x_n - P_s(u)\|^2 + 2\theta_n^2 (\|x_n - P_s(u)\|^2 + \|x_{n-1} - P_s(u)\|^2) \\ &\quad + 2\theta_n \|x_n - P_s(u)\|^2 + \theta_n (\|x_n - P_s(u)\|^2 + \|x_{n-1} - P_s(u)\|^2) \\ &= (1 + 2\theta_n^2 + 3\theta_n) \|x_n - P_s(u)\|^2 + (2\theta_n^2 + \theta_n) \|x_{n-1} - P_s(u)\|^2 \\ &= a_n + \theta_n(2\theta_n + 3) a_n + \theta_n(2\theta_n + 1) a_{n-1}. \end{aligned}$$

From (3.4), we have $\|z_n - P_S(u)\| \le \|y_n - P_S(u)\|$, so:

$$\begin{aligned} a_{n+1} = \|x_{n+1} - P_S(u)\|^2 &= \|\alpha_n(u - P_S(u)) + (1-\alpha_n)(z_n - P_S(u))\|^2 \\ &= \alpha_n^2 \|u - P_S(u)\|^2 + (1-\alpha_n)^2 \|z_n - P_S(u)\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n) \langle u - P_S(u), z_n - P_S(u) \rangle \\ &\le \alpha_n^2 \|u - P_S(u)\|^2 + (1-\alpha_n)^2 [a_n + \theta_n(2\theta_n + 3) a_n + \theta_n(2\theta_n + 1) a_{n-1}] \\ &\quad + 2\alpha_n(1-\alpha_n) \langle u - P_S(u), z_n - P_S(u) \rangle. \end{aligned}$$

Let:

$$b_n = \frac{(1-\alpha_n)^2 \theta_n(2\theta_n + 3) a_n + (1-\alpha_n)^2 \theta_n(2\theta_n + 1) a_{n-1} + \alpha_n^2 \|u - P_s(u)\|^2 + 2\alpha_n(1-\alpha_n) \langle u - P_s(u),\, z_n - P_s(u) \rangle}{\alpha_n}.$$

We have

$$a_{n+1} \le (1-\alpha_n) a_n + \alpha_n b_n.$$

By Theorem 3.1, $z_n \rightharpoonup x^*$, where $x^* \in S$. By Lemma 2.1 we have:

$$\lim_{n\to\infty} \langle u - P_S(u), z_n - P_S(u)\rangle = \langle u - P_S(u), x^* - P_S(u)\rangle \le 0, \quad \forall x^* \in S.$$

From (3.3), we have

$$\|z_n - c\| \le \|y_n - c\| (\forall c \in S). \tag{3.5}$$

Since

$$\begin{aligned} \|y_n - c\| &= \|x_n - c + \theta_n(x_n - x_{n-1})\| \\ &\le \|x_n - c\| + \theta_n\|x_n - x_{n-1}\| \\ &= \|x_n - c\| + \theta_n\|x_n - c + c - x_{n-1}\| \\ &\le \|x_n - c\| + \theta_n\|x_n - c\| + \theta_n\|x_{n-1} - c\| \\ &= (1+\theta_n)\|x_n - c\| + \theta_n\|x_{n-1} - c\|. \end{aligned} \tag{3.6}$$

Let $\dfrac{\theta_n}{\alpha_n} = \beta_n$.Since

$$\begin{aligned} &\|x_{n+1} - c\| = \|\alpha_n(u-c) + (1-\alpha_n)(z_n - c)\| \\ &\le \alpha_n\|u-c\| + (1-\alpha_n)\|z_n - c\| \\ &\le \alpha_n\|u-c\| + (1-\alpha_n)\|y_n - c\| \\ &\le \alpha_n\|u-c\| + (1-\alpha_n)[(1+\theta_n)\|x_n - c\| + \theta_n\|x_{n-1} - c\|] \\ &= \alpha_n\|u-c\| + (1-\alpha_n)(1+\alpha_n\beta_n)\|x_n - c\| + (1-\alpha_n)\alpha_n\beta_n\|x_{n-1} - c\| \\ &= [1 - \alpha_n(1 - \beta_n + \alpha_n\beta_n)]\|x_n - c\| + (1-\alpha_n)\alpha_n\beta_n\|x_{n-1} - c\| + \alpha_n\|u-c\|. \end{aligned}$$

By the given conditions $\alpha_n \subset (0,1)$, $\alpha_n \to 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\beta_n = 0$, using Lemma 3.2, we find that $\{\|x_n - c\|\}$ is bounded. Thus, combine with (3.5)(3.6),the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ are all bounded.

Note that $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n} = \lim_{n\to\infty}\alpha_n = 0$, so $\overline{\lim}_{n\to\infty} b_n \le 0$. Therefore, by Lemma 2.2, we have $\lim_{n\to\infty} a_n = 0$, hence:

$$x_n \to P_S(u), \quad (n\to\infty).$$

This completes the proof of strong convergence.

Remark 3.1 It is noteworthy that the sequences $\{x_n\}$ generated by Algorithms 3.1 and 3.2 in Reference [3], which correspond to Equations (1.3) and (1.4) in the present work, are Fejér monotonic with respect to the set C. In contrast, the sequences $\{x_n\}$ produced by Algorithms 3.1 and 3.2 proposed herein fail to exhibit Fejér monotonicity with respect to C. Consequently, the methodologies adopted for proving the four main theorems in this paper differ fundamentally from those employed in Reference [3].

4. Weak and strong convergence of iterative algorithms with variable step size

Algorithm 4.1. Choose an arbitrary initial point $x_0 \in \mathscr{H}$, and set $x_1 = x_0$. Given the current iteration x_n, compute the next iteration x_{n+1} as follows:

$$\begin{cases} y_n &= x_n + \beta_n(x_n - x_{n-1}), \\ x_{n+1} &= y_n - \tau_n \sum\limits_{i=0}^{N} A_i^*(A_i y_n - T_i(A_i y_n)). \end{cases}$$

$\tau_n = \dfrac{k\sum\limits_{i=0}^{N}\|A_i y_n - T_i(A_i y_n)\|^2}{2\left\|\sum\limits_{i=0}^{N} A_i^*(I-T_i)A_i y_n\right\|^2}$. If $\|\sum_{i=0}^{N} A_i^*(I-T_i)A_i y_n\| = 0$, then the iteration stops.

Theorem 4.1. *Let T_i be k_i-semi-contractive mappings ($\forall i \in Q$) satisfying the demi-closed principle. If $\sum_{n=0}^{\infty} \beta_n < +\infty$, then the sequence $\{x_n\}$ generated by Algorithm 4.1 weakly converges to a solution of problem (1.2).*

Proof. From Algorithm 4.1, we have:

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \left\| (y_n - z) - \tau_n \sum_{i=0}^{N} A_i^* (A_i y_n - T_i(A_i y_n)) \right\|^2 \\ &= \|y_n - z\|^2 + \tau_n^2 \left\| \sum_{i=0}^{N} A_i^* (A_i y_n - T_i(A_i y_n)) \right\|^2 \\ &\quad - 2\tau_n \sum_{i=0}^{N} \langle y_n - z, A_i^* (A_i y_n - T_i(A_i y_n)) \rangle . \end{aligned} \tag{4.1}$$

Since T_i is a k_i-semi-contractive mapping, by Definition 2.2, we have:

$$\begin{aligned} &\langle A_i y_n - A_i z, A_i y_n - T_i(A_i y_n) \rangle \geq \frac{1-k_i}{2} \|A_i y_n - T_i(A_i y_n)\|^2 \\ &\geq \frac{k}{2} \|A_i y_n - T_i(A_i y_n)\|^2 . \end{aligned}$$

Substituting into (4.1), we get:

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq \|y_n - z\|^2 + \tau_n^2 \left\| \sum_{i=0}^{N} A_i^* (A_i y_n - T_i(A_i y_n)) \right\|^2 - \tau_n k \sum_{i=0}^{N} \|A_i y_n - T_i(A_i y_n)\|^2 \\ &= \|y_n - z\|^2 + \tau_n \frac{k \sum_{i=0}^{N} \|A_i y_n - T_i(A_i y_n)\|^2}{2 \left\| \sum_{i=0}^{N} A_i^* (I - T_i) A_i y_n \right\|^2} \left\| \sum_{i=0}^{N} A_i^* (A_i y_n - T_i(A_i y_n)) \right\|^2 \\ &\quad - \tau_n k \sum_{i=0}^{N} \|A_i y_n - T_i(A_i y_n)\|^2 \\ &= \|y_n - z\|^2 - \frac{\tau_n k}{2} \sum_{i=0}^{N} \|A_i y_n - T_i(A_i y_n)\|^2 \leq \|y_n - z\|^2 . \end{aligned} \tag{4.2}$$

So $\|x_{n+1} - z\| \leq \|y_n - z\|$. From the iteration format $y_n = x_n + \beta_n (x_n - x_{n-1})$, we get:

$$\begin{aligned} \|y_n - z\| &= \|x_n - z + \beta_n (x_n - x_{n-1})\| \\ &\leq \|x_n - z\| + \beta_n \|(x_n - z) - (x_{n-1} - z)\| \\ &\leq \|x_n - z\| + \beta_n \|x_n - z\| + \beta_n \|x_{n-1} - z\| \\ &= (1 + \beta_n) \|x_n - z\| + \beta_n \|x_{n-1} - z\| . \end{aligned} \tag{4.3}$$

Hence, $\|x_{n+1} - z\| \leq (1+\beta_n)\|x_n - z\| + \beta_n \|x_{n-1} - z\|$. Since $\sum_{n=0}^{\infty} \beta_n < \infty$, using the above inequality and Lemma 3.1, combined with the proof process of Theorem 3.1, we find that the sequence $\{\|x_n - z\|\}$ is bounded. Thus, from (4.3), $\{y_n\}$ is also bounded, and $\{A_i y_n\}$, $\{T_i(A_i y_n)\}$ are all bounded. Therefore, there exists $G > 0$ such that:

$$0 \leq \max\{\|x_n - z\|, \|x_n\|, \|y_n\|, \|A_i y_n\|, \|T_i(A_i y_n)\|\} \leq G, \; n = 0, 1, \ldots i = 0, 1, \ldots N .$$

Similar to the treatment in Theorem 3.1, for $n_0 \in \mathbb{N}^+$, $\forall n \geq n_0$, we have:

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 + 8G^2 \beta_n .$$

Combining with (4.2), we have:

$$\|x_{n+1}-z\|^2 \le \|x_n-z\|^2 - \frac{\tau_n k}{2}\sum_{i=0}^{N}\|A_iy_n - T_i(A_iy_n)\|^2 + 8G^2\beta_n, (n = n_0, n_0+1, ...).$$

Thus:

$$\frac{\tau_n k}{2}\sum_{i=0}^{N}\|A_iy_n - T_i(A_iy_n)\|^2 \le \|x_n-z\|^2 - \|x_{n+1}-z\|^2 + 8G^2\beta_n.$$

By the Cauchy-Schwarz inequality:

$$\begin{aligned}
\tau_n &= \frac{k\sum_{i=0}^{N}\|A_iy_n - T_i(A_iy_n)\|^2}{2\left\|\sum_{i=0}^{N}A_i^*(I-T_i)A_iy_n\right\|^2} \\
&\ge \frac{k\sum_{i=0}^{N}\|A_iy_n - T_i(A_iy_n)\|^2}{2\left(\sum_{i=0}^{N}\|A_i^*(I-T_i)A_iy_n\|\right)^2} \\
&\ge \frac{k\sum_{i=0}^{N}\|A_iy_n - T_i(A_iy_n)\|^2}{2\sum_{i=0}^{N}\left(\|A_i^*\|\cdot\|A_iy_n - T_i(A_iy_n)\|\right)^2} \\
&\ge \frac{k\sum_{i=0}^{N}\|A_iy_n - T_i(A_iy_n)\|^2}{2\left(\sum_{i=0}^{N}\|A_i^*\|^2\right)\cdot\left(\sum_{i=0}^{N}\|A_iy_n - T_i(A_iy_n)\|^2\right)} \\
&= \frac{k}{2\sum_{i=0}^{N}\|A_i^*\|^2} > 0.
\end{aligned}$$

Hence:

$$\begin{aligned}
\frac{k^2}{4\sum_{i=0}^{N}\|A_i^*\|^2}\sum_{i=0}^{N}\|A_iy_n - T_i(A_iy_n)\|^2 &\le \frac{\tau_n k}{2}\sum_{i=0}^{N}\|A_iy_n - T_i(A_iy_n)\|^2 \\
&\le \|x_n-z\|^2 - \|x_{n+1}-z\|^2 + 8G^2\beta_n.
\end{aligned}$$

Then,we get

$$\begin{aligned}
\frac{k^2}{4\sum_{i=0}^{N}\|A_i^*\|^2}\sum_{n=n_0}^{N}\left(\sum_{i=0}^{N}\|A_iy_n - T_i(A_iy_n)\|^2\right) &\le \sum_{n=n_0}^{N}\left(\|x_n-z\|^2 - \|x_{n+1}-z\|^2\right) + \sum_{n=n_0}^{N}8G^2\beta_n \\
&= \|x_{n_0}-z\|^2 - \lim_{n\to\infty}\|x_{n+1}-z\|^2 + \sum_{n=n_0}^{N}8G^2\beta_n \\
&\le \|x_{n_0}-z\|^2 + \sum_{n=n_0}^{N}8G^2\beta_n < +\infty.
\end{aligned}$$

This implies that the series $\sum_{n=n_0}^{N}(\sum_{i=0}^{N}\|A_iy_n - T_i(A_iy_n)\|^2$ converges, so $A_iy_n = T_i(A_iy_n), \forall i \in \eta$.

Similar to the proof of Theorem 3.1, we can show that $A_ix_n - T_i(A_ix_n) \to 0 (n \to \infty)$, and subsequently, as in Theorem 3.1, we can prove that $x_n \rightharpoonup x^* \in S$.

Algorithm 4.2. Choose an arbitrary initial point $x_0 \in \mathscr{H}$ and $u \in \mathscr{H}$, and set $x_1 = x_0$. Given the current iteration x_n, compute the next iteration x_{n+1} as follows:

$$\begin{cases}
y_n &= x_n + \beta_n(x_n - x_{n-1}), \\
z_n &= y_n - \tau_n \sum_{i=0}^{N} A_i^*(A_iy_n - T_i(A_iy_n)), \\
x_{n+1} &= \alpha_n u + (1-\alpha_n)z_n.
\end{cases}$$

$\tau_n = \frac{k\sum_{i=0}^{N}\|A_iy_n - T_i(A_iy_n)\|^2}{2\|\sum_{i=0}^{N}A_i^*(I-T_i)A_iy_n\|^2}$.If $\|\sum_{i=0}^{N}A_i^*(I-T_i)A_iy_n\| = 0$, then the iteration stops.

Theorem 4.2. *Let T_i be k_i-semi-contractive mappings $(\forall i \in Q)$ satisfying the demi-closed principle. If $\{\alpha_n\} \subset (0,1)$ with $\alpha_n \to 0 (n \to +\infty)$, $\sum_{n=0}^{\infty} \alpha_n = +\infty$, $\frac{\beta_n}{\alpha_n} \to 0 (n \to +\infty)$, and $\sum_{n=0}^{\infty} \beta_n < +\infty$, then $x_n \to P_S(u)(n \to +\infty)$.*

Proof. The specific proof process refers to Theorem 3.2 and is omitted here.

5. Numerical experiments

To verify the effectiveness of the proposed algorithms, we conduct numerical experiments comparing our inertial algorithms with the original methods from [3].

5.1. Problem setting

Find $x^* \in \mathscr{H}$ such that:

$$x^* \in F(T_0) \cap A_1^{-1}(F(T_1)) \cap A_2^{-1}(F(T_2)) \cap A_3^{-1}(F(T_3)).$$

Experimental Setup:

- Main space: $\mathscr{H} \equiv \mathbb{R}^{10}$
- Output space: $\mathscr{H}_i = \mathbb{R}^8$
- Number of operators: $N = 3$

Specific operators:

- k-semi-contractive mapping on main space: $T_0 : \mathbb{R}^{10} \to \mathbb{R}^{10}$, $T_0(x) = 0.7x + 0.3\frac{e^{0.5x} - e^{-0.5x}}{e^{0.5x} + e^{-0.5x}}$ $(k_0 = 0.3)$
- Semi-contractive mappings on output space:

$$T_1(y) = 0.8y + 0.2\arctan(0.6y), \quad (k_1 = 0.2).$$
$$T_2(y) = 0.75y + 0.25(y/(1+|y|)), (k_2 = 0.25).$$
$$T_3(y) = 0.85y + 0.15\exp(-0.4|y|), (k_3 = 0.3).$$

Bounded linear operators:

$$A_i : \mathbb{R}^{10} \to \mathbb{R}^8.$$
$$A_1 = \frac{\text{eye}(8,10) + 0.1 \times R_1}{3.2}, \quad \|A_1\| = 0.312.$$
$$A_2 = \frac{\text{eye}(8,10) + 0.1 \times R_2}{3.0}, \quad \|A_2\| = 0.333.$$
$$A_3 = \frac{\text{eye}(8,10) + 0.1 \times R_3}{2.8}, \quad \|A_3\| = 0.357.$$

where $\text{eye}(m,n)$ denotes the $m \times n$ identity matrix, and R_1, R_2, R_3 are randomly generated matrices.

5.2. Algorithm comparison

Algorithm 4.1 from [3] (without inertia):

$$x_{n+1} = x_n - \tau \sum_{i=0}^{3} A_i^* (\mathscr{A}_i x_n - T_i(\mathscr{A}_i x_n)).$$

Where

$$\tau = \frac{\lambda \sum_{i=0}^{3} \|\mathscr{A}_i y_n - T_i(\mathscr{A}_i y_n)\|^2}{2\left\|\sum_{i=0}^{3} A_i^* (I - T_i)\mathscr{A}_i x_n\right\|^2}, \quad k = 0.7.$$

Our improved Algorithm 4.1 (with inertia):

$$\begin{cases} y_n = x_n + \beta_n (x_n - x_{n-1}), \\ x_{n+1} = y_n - \tau_n \sum_{i=0}^{3} A_i^* (A_i y_n - T_i(A_i y_n)). \end{cases}$$

Where

$$\tau_n = \frac{k \sum\limits_{i=0}^{3} \|A_i y_n - T_i(A_i y_n)\|^2}{2\left\|\sum\limits_{i=0}^{3} A_i^* (I - T_i) A_i y_n\right\|^2}, \quad k = 0.7, \quad \beta_n = \frac{0.3}{(1 + 0.05n)^{0.6}}.$$

Algorithm 4.2 from [3] (without inertia):

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)\left[x_n - \tau_n \sum_{i=0}^{3} A_i^* (A_i x_n - T_i(A_i x_n))\right],$$

Where

$$\alpha_n = \frac{1}{(1+n)^{0.8}}, \quad \tau_n = \frac{k \sum\limits_{i=0}^{3} \|A_i x_n - T_i(A_i x_n)\|^2}{2\left\|\sum\limits_{i=0}^{3} A_i^* (I - T_i) A_i x_n\right\|^2}, \quad k = 0.7.$$

Our improved Algorithm 4.2 (with inertia):

$$\begin{cases} y_n = x_n + \beta_n (x_n - x_{n-1}), \\ z_n = y_n - \tau_n \sum_{i=0}^{3} A_i^* (A_i y_n - T_i(A_i y_n)), \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n. \end{cases}$$

Where

$$\alpha_n = \frac{1}{(1+n)^{0.8}}, \quad \beta_n = \frac{0.3}{(1 + 0.05n)^{0.5}}, \quad \tau_n = \frac{k \sum\limits_{i=0}^{3} \|A_i y_n - T_i(A_i y_n)\|^2}{2\left\|\sum\limits_{i=0}^{3} A_i^* (I - T_i) A_i y_n\right\|^2}, \quad k = 0.7.$$

5.3. *Algorithm comparison figures*

The numerical results obtained using Matlab R2020b demonstrate that the inclusion of the inertial term significantly accelerates the convergence. Furthermore, the error curve exhibits a smooth and monotonic decrease without oscillations.

Figure 1 shows a comparison of Algorithm 4.1. The non-inertial algorithm reaches convergence criteria after approximately 60 iterations. In contrast, the inertial algorithm achieves the same accuracy with only about 52 iterations, a reduction of approximately 13.3% in iteration count. This indicates that the inertial term effectively corrects the search direction by incorporating momentum information from historical iteration points, thereby approximating the solution set with fewer steps. Within the same computation time, the error reduction rate of the inertial algorithm is consistently faster than that of the non-inertial algorithm. The non-inertial algorithm takes approximately 0.028 seconds to converge, while the inertial algorithm takes only about 0.020 seconds, a reduction in computation time of approximately 28.6%. Figure 2 shows a comparison of Algorithm 4.2. The improved algorithm achieves convergence criteria 0.013 seconds earlier, demonstrating a significant time efficiency advantage.

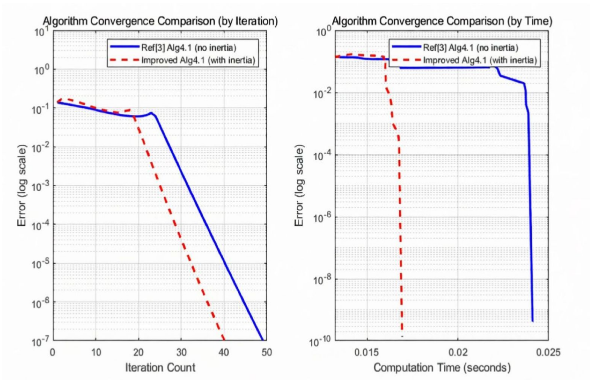

Figure 1: Algorithm 4.1 comparison

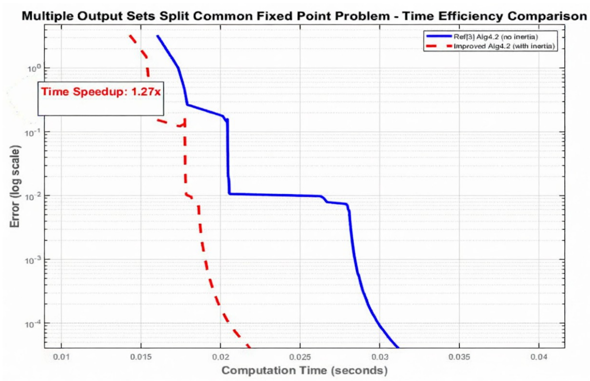

Figure 2: Algorithm 4.2 comparison

5.4. *Algorithm robustness analysis*

To investigate the robustness of the algorithm to the setting of the inertial parameter β_n, we examined the performance variation when the decay coefficient γ fluctuates within the interval $[0.3, 0.8]$. The inertial parameter was set as $\beta_n = 0.3/(1+0.05n)^\gamma$, with γ taking values from the set $\{0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$. As shown in Figures 3 and 4,

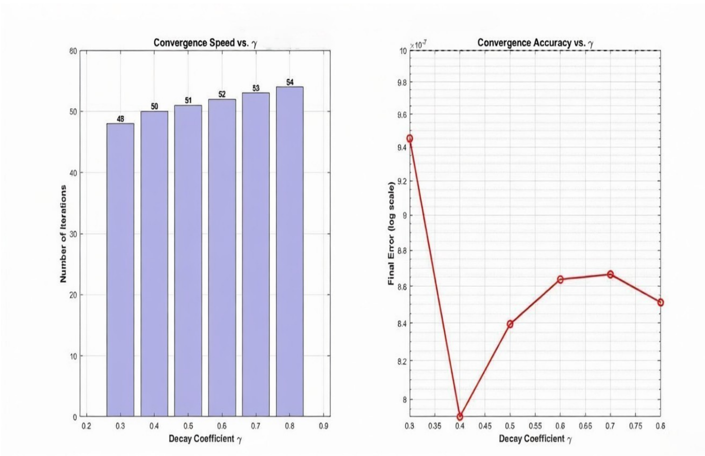

Figure 3: Algorithm4.1 robustness analysis

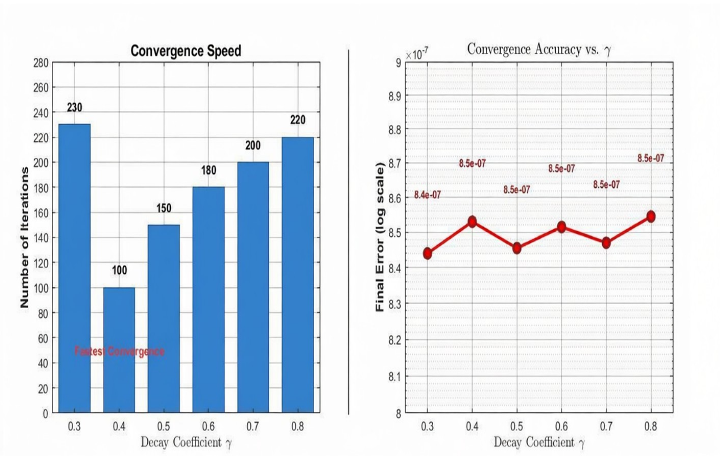

Figure 4: Algorithm4.2 robustness analysis

when γ varies within a relatively large range from 0.3 to 0.8, the algorithm consistently exhibits stable convergence. The number of iterations changes smoothly without any divergence or sharp degradation in convergence speed. Even at the boundary values $\gamma = 0.3$ and $\gamma = 0.8$, the algorithm still converges successfully. This demonstrates that the proposed inertial algorithm is insensitive to the parameter γ setting, indicating excellent parameter robustness. In practical applications, stable acceleration effects can be achieved without precise parameter tuning.

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Use of AI Tools

AI tools were not employed in generating, analyzing, or interpreting the results.

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