J. Inequal. Math. Anal., 2(1), 2026, 31–39

Journal of Inequalities and Mathematical Analysis

ISSN:3062-3251

Journal Homepage: www.inequalmath.com

Research Article

Several New Integral Inequalities of Hermite–Hadamard Type for Extended $\phi_{h\text{-}s}$-convex Functions

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Abstract

In the paper, the authors modify the definitions of $\phi_{h\text{-}s}$-convex functions and extended $\phi_{h\text{-}s}$-convex functions, establish two new integral identities, and, by virtue of these two integral identities, present several new integral inequalities of the Hermite–Hadamard type for extended $\phi_{h\text{-}s}$-convex functions.

Keywords: Integral inequality, Hermite–Hadamard type inequality, extended $\phi_{h\text{-}s}$-convex function, Hölder inequality.

2020 MSC: 26A51, 26D15, 41A55.

1. Introduction

In this paper, let $I \subseteq \mathbb{R} = (-\infty, \infty)$ stand for an interval.

In the 2016 paper [1], the authors introduced the following concept of $\phi_{h\text{-}s}$-convex functions and studied some properties of $\phi_{h\text{-}s}$-convex functions. See also [2].

Definition 1.1 (Essentially modified version of [1, Definition 2.1])**.** Let $h : (0,1) \to \mathbb{R}_+ = (0,\infty)$ and $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ such that $\phi(I) = \{\phi(x), x \in I\}$ is an interval with inner points. For $s \in [0,1]$, a function $f : \phi(I) \to \mathbb{R}$ is said to be $\phi_{h\text{-}s}$-convex on $\phi(I)$ if the inequality

$$f(t\phi(x) + (1-t)\phi(y)) \le \left[\frac{t}{h(t)}\right]^s f(\phi(x)) + \left[\frac{1-t}{h(1-t)}\right]^s f(\phi(y)) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in (0,1)$.

Remark 1.1. Under the condition of Definition 1.1,

1. if $s \in (0,1]$, $\phi(x) = x$ for $x \in I$, and $h(t) = 1$ for $t \in (0,1)$, then the $\phi_{h\text{-}s}$-convex function becomes an s-convex function, see [3, 4];

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doi: 10.63286/jima.036

Received: 26 September 2025 Revised: 2 January 2026 Accepted: 5 January 2026

2. if $s \in (0,1]$, $\phi(x) = x$ for $x \in I$, and $h(t) = \frac{1}{t^2}$ for $t \in (0,1)$, then the $\phi_{h\text{-}s}$-convex function is an s-Godunova-Levin convex function, see [5];
3. if $s = 1$, $\phi(x) = x$ for $x \in I$, and $h(t) = 2\sqrt{t(1-t)}$ for $t \in (0,1)$, then the $\phi_{h\text{-}s}$-convex function is an MT-convex function, see [6];
4. if $s = 1$, $\phi(x) = x$ for $x \in I$, and $h(t) = \frac{t}{h_1(t)}$ for $t \in (0,1)$, then the $\phi_{h\text{-}s}$-convex function is an h-convex function defined by [7, p. 304, Definition 4].

In [8], the concept of extended s-convex functions was innovated below.

Definition 1.2 ([8, Definition 2.1])**.** A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be extended s-convex on an interval I if

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y) \tag{1.2}$$

holds for all $x, y \in I$ and $t \in (0,1)$ and for some fixed $s \in [-1,1]$.

In [9], the authors extended $s \in [0,1]$ in Definition 1.1 to $s \in [-1,1]$ and improved Definition 1.1 by introducing the following extended $\phi_{h\text{-}s}$-convex functions.

Definition 1.3 (Slightly modified version of [9, Definition 2.1])**.** Let $h : (0,1) \to \mathbb{R}_+$ and $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ such that $\phi(I) = \{\phi(x), x \in I\}$ is an interval with inner points. For some $s \in [-1,1]$, a function $f : \phi(I) \to \mathbb{R}$ is said to be extended $\phi_{h\text{-}s}$-convex on $\phi(I)$ if the inequality (1.1) holds for all $x, y \in I$ and $t \in (0,1)$.

Remark 1.2. Under the condition of Definition 1.3,

1. if $s \in [0,1]$, then the extended $\phi_{h\text{-}s}$-convex function is a $\phi_{h\text{-}s}$-convex function, see [1];
2. if $\phi(x) = x$ for $x \in I$ and $h(t) = 1$ for $t \in (0,1)$, then the extended $\phi_{h\text{-}s}$-function is an extended s-convex function, see [8].

Example 1. For some fixed $s \in [-1,1] \setminus \{0\}$ and $0 < p < 1$, define $f(x) = x^{p/(p+1)}$ for $x \in (0,1]$, $h(t) = t^{1-p/s(p+1)}$ for $t \in (0,1)$, and

$$\phi(x) = \begin{cases} x^{p+1}, & x \in (0,1) \setminus \left\{\dfrac{1}{2}\right\}; \\[2ex] 1, & x = \dfrac{1}{2}; \\[2ex] \left(\dfrac{1}{2}\right)^{p+1}, & x = 1. \end{cases}$$

Then $f(\phi(x)) = x^p$ and $\left[\frac{t}{h(t)}\right]^s = t^{p/(p+1)}$ for $x \in (0,1) \setminus \{\frac{1}{2}\}$ and $t \in (0,1)$, with $f(\phi(\frac{1}{2})) = 1$ and $f(\phi(1)) = (\frac{1}{2})^p$.

Using the inequality $(u+1)^r \le u^r + 1$ for $u > 0$ and $0 < r < 1$, for all $x, y \in (0,1]$ and $t \in (0,1)$, we obtain

$$\left[tx^{p+1} + (1-t)y^{p+1}\right]^{p/(p+1)} \le t^{p/(p+1)}x^p + (1-t)^{p/(p+1)}y^p.$$

This means that, although ϕ is not continuous, but $\phi(I) = (0,1]$ is an interval with inner points, the function $f(x) = x^{p/(p+1)}$ is still extended $\phi_{h\text{-}s}$-convex on $(0,1]$.

Example 2. When $\phi(I) = \{2,5\}$, which is not an interval with inner points, the function f in (1.1) is defined on the set $\phi(I) = \{2,5\}$ with two points. But the left hand side in (1.1) requires f to take values in the interval $(2,5) = \{2t + 5(1-t), t \in (0,1)\} = \{5 - 3t, t \in (0,1)\}$. This is the reason why we required $\phi(I)$ in Definitions 1.1 and 1.3 to be an interval with inner points.

Example 3. For $0 < p < 1$, let

$$\phi_1(x) = 2x^{p+1}, \quad x \in I_1 = \left(0, 2^{-1/(p+1)}\right]$$

and

$$\phi_2(x) = x^{p+1}, \quad x \in I_2 = (0,1].$$

It is easy to see that $\phi_1(I_1) = \phi_2(I_2) = (0,1]$. If a function f were extended ${\phi_1}_{h\text{-}s}$-convex on $\phi_1(I_1) = (0,1]$ and extended ${\phi_2}_{h\text{-}s}$-convex on $\phi_2(I_2) = (0,1]$, we can still differentiate them naturally.

In [9], some integral inequalities for extended $\phi_{h\text{-}s}$-convex functions were established. We now modify and restate Theorems 3.1 to 3.4 without any change of their proofs in [9] as follows.

Theorem 1.1 ([9, Theorem 3.1]). *Let $h:(0,1)\to\mathbb{R}_+$, let $\phi: I\subseteq\mathbb{R}\to\mathbb{R}$ and $f:\phi(I)\to\mathbb{R}$ be differentiable, and let $|\phi'|$ be convex. For some $s\in[-1,1]$ and $q\ge1$, if $|f'|^q$ is an increasing extended $\phi_{h\text{-}s}$-convex function on $\phi(I)$, $f'(\phi)\phi'\in L_1(I)$, and $(1-2x)\left[\frac{x}{h(x)}\right]^s\in L_1([0,1])$, then for $a,b\in I^\circ$ with $a<b$, we have*

$$\left|\frac{f(\phi(a))+f(\phi(b))}{2}-\frac{1}{b-a}\int_a^b f(\phi(x))\,\mathrm{d}x\right| \le \frac{(b-a)\|\phi'\|_\infty}{2^{2-1/q}}\left(\left(|f'(\phi(a))|^q+|f'(\phi(b))|^q\right)\int_0^1|1-2t|\left[\frac{t}{h(t)}\right]^s\mathrm{d}t\right)^{1/q},$$

where $\|\phi'\|_\infty=\sup_{x\in[a,b]}|\phi'(x)|$.

Theorem 1.2 ([9, Theorem 3.2]). *Let $h:(0,1)\to\mathbb{R}_+$, let $\phi: I\subseteq\mathbb{R}\to\mathbb{R}$ and $f:\phi(I)\to\mathbb{R}$ be differentiable, and let $|\phi'|$ be convex. For some $s\in[-1,1]$ and $q>1$, if $|f'|^q$ is an increasing extended $\phi_{h\text{-}s}$-convex function on $\phi(I)$, $f'(\phi)\phi'\in L_1(I)$, and $\left[\frac{x}{h(x)}\right]^s\in L_1([0,1])$, then for $a,b\in I^\circ$ with $a<b$, we have*

$$\left|\frac{f(\phi(a))+f(\phi(b))}{2}-\frac{1}{b-a}\int_a^b f(\phi(x))\,\mathrm{d}x\right| \le \frac{(b-a)\|\phi'\|_\infty}{2}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\left[|f'(\phi(a))|^q+|f'(\phi(b))|^q\right]\int_0^1\left[\frac{t}{h(t)}\right]^s\mathrm{d}t\right)^{1/q},$$

where $\|\phi'\|_\infty=\sup_{x\in[a,b]}|\phi'(x)|$.

Theorem 1.3 ([9, Theorem 3.3]). *Let $h:(0,1)\to\mathbb{R}_+$, let $\phi: I\subseteq\mathbb{R}\to\mathbb{R}$ and $f:\phi(I)\to\mathbb{R}$ be differentiable, and let $|\phi'|$ be convex. For some $s\in[-1,1]$ and $q\ge1$, if $|f'|^q$ is an increasing extended $\phi_{h\text{-}s}$-convex function on $\phi(I)$, $f'(\phi)\phi'\in L_1(I)$, and $\frac{x^{s+1}}{h^s(x)},\frac{(1-x)x^s}{h^s(x)}\in L_1([0,1])$, then for $a,b\in I^\circ$ with $a<b$, we have*

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(\phi(x))\,\mathrm{d}x-f\left(\phi\left(\frac{a+b}{2}\right)\right)\right| &\le \frac{(b-a)\|\phi'\|_\infty}{2^{3(1-1/q)}}\left[\left(|f'(\phi(a))|^q\int_0^{1/2}t\left[\frac{t}{h(t)}\right]^s\mathrm{d}t+|f'(\phi(b))|^q\int_{1/2}^1(1-t)\left[\frac{t}{h(t)}\right]^s\mathrm{d}t\right)^{1/q}\right.\\ &\quad\left.+\left(|f'(\phi(a))|^q\int_{1/2}^1(1-t)\left[\frac{t}{h(t)}\right]^s\mathrm{d}t+|f'(\phi(b))|^q\int_0^{1/2}t\left[\frac{t}{h(t)}\right]^s\mathrm{d}t\right)^{1/q}\right],\end{aligned}$$

where $\|\phi'\|_\infty=\sup_{x\in[a,b]}|\phi'(x)|$.

Theorem 1.4 ([9, Theorem 3.4]). *Let $h:(0,1)\to\mathbb{R}_+$, let $\phi: I\subseteq\mathbb{R}\to\mathbb{R}$ and $f:\phi(I)\to\mathbb{R}$ be differentiable, and let $|\phi'|$ be convex. For some $s\in[-1,1]$ and $q>1$, if $|f'|^q$ is an increasing extended $\phi_{h\text{-}s}$-convex function on $\phi(I)$, $f'(\phi)\phi'\in L_1(I)$, and $\frac{x^{s+1}}{h^s(x)},\frac{(1-x)x^s}{h^s(x)}\in L_1([0,1])$, then for $a,b\in I^\circ$ with $a<b$, we have*

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(\phi(x))\,\mathrm{d}x-f\left(\phi\left(\frac{a+b}{2}\right)\right)\right| &\le \frac{(b-a)\|\phi'\|_\infty}{2^{(2q-1)/q}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\\ &\quad\times\left\{\left(|f'(\phi(a))|^q\int_0^{1/2}\left[\frac{t}{h(t)}\right]^s\mathrm{d}t+|f'(\phi(b))|^q\int_{1/2}^1\left[\frac{t}{h(t)}\right]^s\mathrm{d}t\right)^{1/q}\right.\end{aligned}$$

$$+\left(|f'(\phi(a))|^q\int_{1/2}^{1}\left[\frac{t}{h(t)}\right]^s \mathrm{d}t+|f'(\phi(b))|^q\int_{0}^{1/2}\left[\frac{t}{h(t)}\right]^s \mathrm{d}t\right)^{1/q}\Bigg\},$$

where $\|\phi'\|_\infty=\sup_{x\in[a,b]}|\phi'(x)|$.

In this paper, we first establish two new integral identities under the condition that the range of the function ϕ is an interval, and then, by virtue of these two new integral identities and Hölder type integral inequalities, present some integral inequalities of the Hermite–Hadamard type for extended $\phi_{h\text{-}s}$-convex functions.

2. Two new integral identities

In this section, we establish two new integral identities.

Lemma 2.1. *Let $\phi: I\subseteq\mathbb{R}\to\mathbb{R}$, let $\phi(I)$ be an interval with inner points, and let $f:\phi(I)\to\mathbb{R}$ be a differentiable function. If $f'\in L_1(\phi(I))$, then*

$$\frac{f(\phi(a))+f(\phi(b))}{2}-\frac{1}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)}f(u)\,\mathrm{d}u=\frac{\phi(b)-\phi(a)}{2}\int_0^1(1-2t)f'(t\phi(a)+(1-t)\phi(b))\,\mathrm{d}t \tag{2.1}$$

and

$$\frac{1}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)}f(u)\,\mathrm{d}u-f\left(\frac{\phi(a)+\phi(b)}{2}\right)$$
$$=[\phi(b)-\phi(a)]\left[\int_0^{1/2}tf'(t\phi(a)+(1-t)\phi(b))\,\mathrm{d}t+\int_{1/2}^1(t-1)f'(t\phi(a)+(1-t)\phi(b))\,\mathrm{d}t\right] \tag{2.2}$$

for $a,b\in I^\circ$ such that $\phi(a)\ne\phi(b)$

Proof. The identity (2.1) can be proved by using [10, Lemma 2.1].

The identity (2.2) follows from [11, Lemma 2.1]. □

3. New integral inequalities of Hermite–Hadamard type

In this section, we present several new integral inequalities of the Hermite–Hadamard type for extended $\phi_{h\text{-}s}$-convex functions.

Theorem 3.1. *Let $h:(0,1)\to\mathbb{R}_+$, let $\phi: I\subseteq\mathbb{R}\to\mathbb{R}$, and let $f:\phi(I)\to\mathbb{R}$ is an extended $\phi_{h\text{-}s}$-convex function on $\phi(I)$ for some fixed $s\in[-1,1]$. If $f\in L_1(\phi(I))$ and $\left[\frac{x}{h(x)}\right]^s\in L_1([0,1])$, then for $a,b\in I^\circ$ such that $\phi(a)\ne\phi(b)$, we have*

$$\frac{[2h(1/2)]^s}{2}f\left(\frac{\phi(a)+\phi(b)}{2}\right)\le\frac{1}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)}f(u)\,\mathrm{d}u\le[f(\phi(a))+f(\phi(b))]\int_0^1\left[\frac{t}{h(t)}\right]^s\mathrm{d}t.$$

Proof. For $t\in(0,1)$, using the extended $\phi_{h\text{-}s}$-convexity of f, we have

$$f\left(\frac{\phi(a)+\phi(b)}{2}\right)=f\left(\frac{t\phi(a)+(1-t)\phi(b)+(1-t)\phi(a)+t\phi(b)}{2}\right)$$
$$\le\left[\frac{1/2}{h(1/2)}\right]^s f(t\phi(a)+(1-t)\phi(b))+\left[\frac{1/2}{h(1/2)}\right]^s f((1-t)\phi(a)+t\phi(b)) \tag{3.1}$$
$$=\frac{f(t\phi(a)+(1-t)\phi(b))+f((1-t)\phi(a)+t\phi(b))}{[2h(1/2)]^s}.$$

Integrating with respect to $t \in (0,1)$ on the very ends of the inequality (3.1) and making the variable transform $u = t\phi(a) + (1-t)\phi(b)$ for $t \in (0,1)$ result in

$$\begin{aligned} f\left(\frac{\phi(a)+\phi(b)}{2}\right) &= \int_0^1 f\left(\frac{t\phi(a)+(1-t)\phi(b)+(1-t)\phi(a)+t\phi(b)}{2}\right)\mathrm{d}t \\ &\le \frac{1}{[2h(1/2)]^s}\int_0^1 [f(t\phi(a)+(1-t)\phi(b))+f((1-t)\phi(a)+t\phi(b))]\,\mathrm{d}t \\ &= \frac{2[2h(1/2)]^{-s}}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)} f(u)\,\mathrm{d}u. \end{aligned}$$

Further, making the variable transform $u = t\phi(a) + (1-t)\phi(b)$ for $t \in (0,1)$ and using the extended $\phi_{h\text{-}s}$-convexity of f lead to

$$\begin{aligned} \frac{1}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)} f(u)\,\mathrm{d}u &= \int_0^1 f(t\phi(a)+(1-t)\phi(b))\,\mathrm{d}t \\ &\le \int_0^1 \left(\left[\frac{t}{h(t)}\right]^s f(\phi(a)) + \left[\frac{1-t}{h(1-t)}\right]^s f(\phi(b))\right)\mathrm{d}t \\ &= [f(\phi(a))+f(\phi(b))]\int_0^1 \left[\frac{t}{h(t)}\right]^s \mathrm{d}t. \end{aligned}$$

Theorem 3.1 is thus proved. □

Corollary 3.1. *Under conditions of Theorem 3.1,*

1. *if $s = 1$, we have*

$$h\left(\frac{1}{2}\right)f\left(\frac{\phi(a)+\phi(b)}{2}\right) \le \frac{1}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)} f(u)\,\mathrm{d}u \le [f(\phi(a))+f(\phi(b))]\int_0^1 \frac{t}{h(t)}\,\mathrm{d}t;$$

2. *if $s = 0$, we have*

$$\frac{1}{2}f\left(\frac{\phi(a)+\phi(b)}{2}\right) \le \frac{1}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)} f(u)\,\mathrm{d}u \le f(\phi(a))+f(\phi(b));$$

3. *if $s = -1$, we have*

$$\frac{1}{4h(1/2)}f\left(\frac{\phi(a)+\phi(b)}{2}\right) \le \frac{1}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)} f(u)\,\mathrm{d}u \le [f(\phi(a))+f(\phi(b))]\int_0^1 \frac{h(t)}{t}\,\mathrm{d}t.$$

Corollary 3.2. *Let $0 < p < 1$ and $a, b \in \mathbb{R}_+$ with $a < b$. Then*

$$\frac{1}{2^{1/(p+1)}}\left(\frac{a^{p+1}+b^{p+1}}{2}\right)^{p/(p+1)} \le \frac{(p+1)(b^{2p+1}-a^{2p+1})}{(2p+1)(b^{p+1}-a^{p+1})} \le \frac{p+1}{2p+1}(a^p+b^p).$$

Proof. Let $\phi(x) = x^{p+1}$, $f(x) = x^{p/(p+1)}$ for $x \in \mathbb{R}_+$, and $h(t) = t^{1-p/s(p+1)}$ for $t \in (0,1)$ and for some fixed $s \in [-1,1] \setminus \{0\}$. By virtue of [9, Example 2.1], we deduce that $f(x) = x^{p/(p+1)}$ is extended $\phi_{h\text{-}s}$-convex on $\mathbb{R}_+$. Since $f(\phi(x)) = x^p$ and $\left[\frac{t}{h(t)}\right]^s = t^{p/(p+1)}$ for $x \in \mathbb{R}_+$ and $t \in (0,1)$, with the help of Theorem 3.1, we arrive at

$$\frac{[2h(1/2)]^s}{2}f\left(\frac{\phi(a)+\phi(b)}{2}\right) = \frac{1}{2^{1/(p+1)}}\left(\frac{a^{p+1}+b^{p+1}}{2}\right)^{p/(p+1)},$$

$$\frac{1}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)} f(u)\,\mathrm{d}u = \frac{(p+1)(b^{2p+1}-a^{2p+1})}{(2p+1)(b^{p+1}-a^{p+1})},$$

and

$$[f(\phi(a))+f(\phi(b))]\int_0^1 \left[\frac{t}{h(t)}\right]^s \mathrm{d}t = \frac{p+1}{2p+1}(a^p+b^p).$$

The proof of Corollary 3.2 is completed. □

Theorem 3.2. *Let $h:(0,1)\to\mathbb{R}_+$, let $\phi: I\subseteq\mathbb{R}\to\mathbb{R}$, and let $f:\phi(I)\to\mathbb{R}$ be differentiable. For some fixed $s\in[-1,1]$, if $f'\in L_1(\phi(I))$ and $|f'|$ is an extended $\phi_{h\text{-}s}$-convex function on $\phi(I)$, then for $a,b\in I^\circ$ such that $\phi(a)\ne\phi(b)$,*

1. *when $\frac{(1-2x)x^s}{h^s(x)}\in L_1([0,1])$, we have*

$$\left|\frac{f(\phi(a))+f(\phi(b))}{2}-\frac{1}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)} f(u)\,\mathrm{d}u\right| \le \frac{|\phi(b)-\phi(a)|}{2}[|f'(\phi(a))|+|f'(\phi(b))|]\int_0^1 |1-2t|\left[\frac{t}{h(t)}\right]^s \mathrm{d}t; \quad (3.2)$$

2. *when $\frac{x^{s+1}}{h^s(x)}\in L_1\big(\big[0,\frac12\big]\big)$ and $\frac{(1-x)x^s}{h^s(x)}\in L_1\big(\big[\frac12,1\big]\big)$, we have*

$$\left|\frac{1}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)} f(u)\,\mathrm{d}u - f\left(\frac{\phi(a)+\phi(b)}{2}\right)\right| \le |\phi(b)-\phi(a)|[|f'(\phi(a))|+|f'(\phi(b))|]\left(\int_0^{1/2} t\left[\frac{t}{h(t)}\right]^s \mathrm{d}t + \int_{1/2}^1 (1-t)\left[\frac{t}{h(t)}\right]^s \mathrm{d}t\right). \quad (3.3)$$

Proof. Since $|f'|$ is a $\phi_{h\text{-}s}$-convex function, we acquire

$$\left|f'(t\phi(x)+(1-t)\phi(y))\right| \le \left[\frac{t}{h(t)}\right]^s |f'(\phi(x))| + \left[\frac{1-t}{h(1-t)}\right]^s |f'(\phi(y))| \quad (3.4)$$

for all $\phi(x),\phi(y)\in\phi(I)$ and any $t\in(0,1)$.

From Lemma 2.1 and the inequality (3.4), it follows that

$$\begin{aligned}
&\left|\frac{f(\phi(a))+f(\phi(b))}{2}-\frac{1}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)} f(u)\,\mathrm{d}u\right| \\
&\le \frac{|\phi(b)-\phi(a)|}{2}\int_0^1 |1-2t||f'(t\phi(a)+(1-t)\phi(b))|\,\mathrm{d}t \\
&\le \frac{|\phi(b)-\phi(a)|}{2}\left\{\int_0^1 |1-2t|\left(\left[\frac{t}{h(t)}\right]^s|f'(\phi(a))| + \left[\frac{1-t}{h(1-t)}\right]^s|f'(\phi(b))|\right)\mathrm{d}t\right\} \\
&= \frac{|\phi(b)-\phi(a)|}{2}[|f'(\phi(a))|+|f'(\phi(b))|]\int_0^1 |1-2t|\left[\frac{t}{h(t)}\right]^s \mathrm{d}t.
\end{aligned}$$

The inequality (3.2) is proved.

Utilizing Lemma 2.1 and the inequality (3.4), we gain

$$\left|\frac{1}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)} f(u)\,\mathrm{d}u - f\left(\frac{\phi(a)+\phi(b)}{2}\right)\right|$$

$$\begin{aligned}
&\le |\phi(b)-\phi(a)|\left[\int_0^{1/2} t|f'(t\phi(a)+(1-t)\phi(b))|\,\mathrm{d}t+\int_{1/2}^1(1-t)|f'(t\phi(a)+(1-t)\phi(b))|\,\mathrm{d}t\right]\\
&\le |\phi(b)-\phi(a)|\left\{\int_0^{1/2} t\left(\left[\frac{t}{h(t)}\right]^s|f'(\phi(a))|+\left[\frac{1-t}{h(1-t)}\right]^s\big|f'(\phi(b))\big|\right)\mathrm{d}t\right.\\
&\quad\left.+\int_{1/2}^1(1-t)\left(\left[\frac{t}{h(t)}\right]^s|f'(\phi(a))|+\left[\frac{1-t}{h(1-t)}\right]^s\big|f'(\phi(b))\big|\right)\mathrm{d}t\right\}\\
&=|\phi(b)-\phi(a)|[|f'(\phi(a))|+|f'(\phi(b))|]\left(\int_0^{1/2} t\left[\frac{t}{h(t)}\right]^s\mathrm{d}t+\int_{1/2}^1(1-t)\left[\frac{t}{h(t)}\right]^s\mathrm{d}t\right).
\end{aligned}$$

Thus, the inequality (3.3) is proved. Theorem 3.2 is thus proved. □

Theorem 3.3. *Let $h:(0,1)\to\mathbb{R}_+$, let $\phi: I\subseteq\mathbb{R}\to\mathbb{R}$, and let $f:\phi(I)\to\mathbb{R}$ be differentiable. For some fixed $s\in[-1,1]$, $q>1$, and $q>r\ge 0$, if $|f'|^q$ is an extended $\phi_{h\text{-}s}$-convex function on $\phi(I)$, $f'\in L_1(\phi(I))$, and $\frac{|1-2x|^r x^s}{[h(x)]^s}\in L_1([0,1])$, then for $a,b\in I^\circ$ such that $\phi(a)\ne\phi(b)$, we have*

$$\left|\frac{f(\phi(a))+f(\phi(b))}{2}-\frac{1}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)} f(u)\,\mathrm{d}u\right|\le\frac{|\phi(b)-\phi(a)|}{2}\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}\left([|f'(\phi(a))|^q+|f'(\phi(b))|^q]\int_0^1|1-2t|^r\left[\frac{t}{h(t)}\right]^s\mathrm{d}t\right)^{1/q}.$$

Proof. As in the proof of Theorem 3.2, using Lemma 2.1, the inequality (3.4), and Hölder's integral inequality, we obtain

$$\begin{aligned}
&\left|\frac{f(\phi(a))+f(\phi(b))}{2}-\frac{1}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)} f(u)\,\mathrm{d}u\right|\\
&\le\frac{|\phi(b)-\phi(a)|}{2}\int_0^1|1-2t||f'(t\phi(a)+(1-t)\phi(b))|\,\mathrm{d}t\\
&\le\frac{|\phi(b)-\phi(a)|}{2}\left(\int_0^1|1-2t|^{(q-r)/(q-1)}\,\mathrm{d}t\right)^{1-1/q}\left(\int_0^1|1-2t|^r|f'(t\phi(a)+(1-t)\phi(b))|^q\,\mathrm{d}t\right)^{1/q}\\
&\le\frac{|\phi(b)-\phi(a)|}{2}\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}\left\{\int_0^1|1-2t|^r\left(\left[\frac{t}{h(t)}\right]^s|f'(\phi(a))|^q+\left[\frac{1-t}{h(1-t)}\right]^s|f'(\phi(b))|^q\right)\mathrm{d}t\right\}^{1/q}\\
&=\frac{|\phi(b)-\phi(a)|}{2}\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}\left([|f'(\phi(a))|^q+|f'(\phi(b))|^q]\int_0^1|1-2t|^r\left[\frac{t}{h(t)}\right]^s\mathrm{d}t\right)^{1/q}.
\end{aligned}$$

The proof of Theorem 3.3 is completed. □

Theorem 3.4. *Let $h:(0,1)\to\mathbb{R}_+$, let $\phi: I\subseteq\mathbb{R}\to\mathbb{R}$, and let $f:\phi(I)\to\mathbb{R}$ be differentiable. For some fixed $s\in[-1,1]$, $q>1$, and $q>r\ge 0$, if $|f'|^q$ is an extended $\phi_{h\text{-}s}$-convex function on $\phi(I)$, $f'\in L_1(\phi(I))$, $\frac{x^{s+r}}{h^s(x)}\in L_1\left(\left[0,\frac12\right]\right)$, and $\frac{x^s(1-x)^r}{h^s(x)}\in L_1\left(\left[\frac12,1\right]\right)$, then for $a,b\in I^\circ$ such that $\phi(a)\ne\phi(b)$, we have*

$$\left|\frac{1}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)} f(u)\,\mathrm{d}u-f\left(\frac{\phi(a)+\phi(b)}{2}\right)\right|\le|\phi(b)-\phi(a)|\left(\frac12\right)^{(2q-r-1)/q}\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}$$

$$\times\left\{\left(|f'(\phi(a))|^q\int_0^{1/2}t^r\left[\frac{t}{h(t)}\right]^s\mathrm{d}t+|f'(\phi(b))|^q\int_{1/2}^1(1-t)^r\left[\frac{t}{h(t)}\right]^s\mathrm{d}t\right)^{1/q}\right.$$
$$\left.+\left(|f'(\phi(a))|^q\int_{1/2}^1(1-t)^r\left[\frac{t}{h(t)}\right]^s\mathrm{d}t+|f'(\phi(b))|^q\int_0^{1/2}t^r\left[\frac{t}{h(t)}\right]^s\mathrm{d}t\right)^{1/q}\right\}.$$

Proof. As argued in the proofs of Theorems 3.2 and 3.3, we deduce

$$\left|\frac{1}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)}f(u)\,\mathrm{d}u-f\left(\frac{\phi(a)+\phi(b)}{2}\right)\right|$$
$$\le|\phi(b)-\phi(a)|\left[\left(\int_0^{1/2}t^{(q-r)/(q-1)}\,\mathrm{d}t\right)^{1-1/q}\left(\int_0^{1/2}t^r|f'(t\phi(a)+(1-t)\phi(b))|^q\,\mathrm{d}t\right)^{1/q}\right.$$
$$\left.+\left(\int_{1/2}^1(1-t)^{(q-r)/(q-1)}\,\mathrm{d}t\right)^{1-1/q}\left(\int_{1/2}^1(1-t)^r|f'(t\phi(a)+(1-t)\phi(b))|^q\,\mathrm{d}t\right)^{1/q}\right]$$
$$\le|\phi(b)-\phi(a)|\left(\frac{1}{2}\right)^{(2q-r-1)/q}\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}\left\{\left[\int_0^{1/2}t^r\left(\left[\frac{t}{h(t)}\right]^s|f'(\phi(a))|^q\right.\right.\right.$$
$$\left.\left.+\left[\frac{1-t}{h(1-t)}\right]^s|f'(\phi(b))|^q\right)\mathrm{d}t\right]^{1/q}$$
$$\left.+\left[\int_{1/2}^1(1-t)^r\left(\left[\frac{t}{h(t)}\right]^s|f'(\phi(a))|^q+\left[\frac{1-t}{h(1-t)}\right]^s|f'(\phi(b))|^q\right)\mathrm{d}t\right]^{1/q}\right\}$$
$$=|\phi(b)-\phi(a)|\left(\frac{1}{2}\right)^{(2q-r-1)/q}\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}\left\{\left(|f'(\phi(a))|^q\int_0^{1/2}t^r\left[\frac{t}{h(t)}\right]^s\mathrm{d}t\right.\right.$$
$$\left.+|f'(\phi(b))|^q\int_{1/2}^1(1-t)^r\left[\frac{t}{h(t)}\right]^s\mathrm{d}t\right)^{1/q}$$
$$\left.+\left(|f'(\phi(a))|^q\int_{1/2}^1(1-t)^r\left[\frac{t}{h(t)}\right]^s\mathrm{d}t+|f'(\phi(b))|^q\int_0^{1/2}t^r\left[\frac{t}{h(t)}\right]^s\mathrm{d}t\right)^{1/q}\right\}.$$

Theorem 3.4 is thus proved. □

4. Remarks

Finally, we give several remarks about related stuffs.

Remark 4.1. Definition 1.3 in this paper is a slightly modification of [9, Definition 2.1].

Remark 4.2. Remark 3.1 in [9] should be corrected as follows:

> If ϕ has a derivative of the first order and $|\phi'|$ is concave, if for $q>1$ the function $|f'|^q$ is decreasing extended $\phi_{h\text{-}s}$-convex on $\phi(I)$, then Theorems 1.1 to 1.4 in this paper still hold.

Remark 4.3. This paper is a slightly modified version of the preprint "B.-Y. Xi and F. Qi, *Several new integral inequalities of Hermite–Hadamard type for extended $\phi_{h\text{-}s}$-convex functions*, Preprints.org, 2025100012, 9 pages. DOI: `https://doi.org/10.20944/preprints202510.0012.v1`."

Funding

The first author was partially supported by the National Natural Science Foundation of China (Grant No. 12361013). The second author was partially supported by the Natural Science Foundation of Inner Mongolia Autonomous Region (Grant No. 2025QN01041) and by the Youth Project of Hulunbuir City for Basic Research and Applied Basic Research (Grant No. GH2024020).

Acknowledgements

The authors appreciate anonymous referees for their careful corrections to the original version of this paper.

Use of AI Tools

AI tools were not employed in generating, analyzing, or interpreting the results.

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