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Research Article

Nontangential Limits and Nontangetial Boundedness for Solutions of Hermite Type Equations

Guillermo J. Flores [a,b]

[a] Universidad Nacional de Córdoba, Facultad de Matemática, Astronomía, Física y Computación, Grupo de Ecuaciones Diferenciales y Análisis Armónico, Córdoba, Argentina.

[b] Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), CIEM, Córdoba, Argentina.

Abstract

Nontangential convergence on the boundary is studied for $\mathfrak{L}$-harmonic functions defined on the upper half-space, where $\mathfrak{L}$ is either a slightly more general operator than the Hermite operator or is a perturbation of the classical Laplacian. In addition, the Poisson kernel and the Poisson integral associated with $\mathfrak{L}$ are analyzed in detail and several properties of independent interest are presented.

Keywords: Calderón theorem, Poisson integral, Hermite operator, Ornstein-Uhlenbeck operator, perturbed laplacian

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1. Introduction

1.1. A Calderón theorem for harmonic functions

It was proved by A. P. Calderón in [1] that nontangential limits and nontangetial boundedness are essentially equivalent for harmonic functions on the Euclidean upper half-space $\mathbb{R}^{1+d}_+ = (0,\infty)\times\mathbb{R}^d$. More specifically, let $u(t,x)$ be a function defined on $\mathbb{R}^{1+d}_+$. For a Lebesgue measurable set E in $\mathbb{R}^d$, u is said to be *nontangentially bounded* at all the points of E, if for each $x_0 \in E$ and ρ positive there is a constant $C_{x_0,\rho}$ such that

$$\sup_{(t,x)\in\Gamma_\rho(x_0)} |u(t,x)| \le C_{x_0,\rho},$$

where $\Gamma_\rho(x_0)$ is denoting the open cone with vertex $(0,x_0)$ and aperture ρ, $\Gamma_\rho(x_0) = \{(t,x)\in\mathbb{R}^{1+d}_+ : |x-x_0| < \rho t\}$. Likewise, u is said to have *nontangential limit* l at x_0 if, for each $\rho > 0$,

$$\lim_{(t,x)\to(0,x_0)} u(t,x) = l,$$

where (t,x) tends to $(0,x_0)$ within $\Gamma_\rho(x_0)$. Then, Calderón theorem states the following.

If u is a harmonic function on $\mathbb{R}^{1+d}_+$ and is nontangentially bounded at all the points of a set E of positive Lebesgue measure, then u has nontangential limit at almost every point of E.

This is a fundamental result to obtain significant extensions of the classical Hardy spaces H^p, see for instance Chapter VI, §4 of [2]. It is also important to highlight that L. Carleson proved a more general version of Calderon theorem in [3].

*Corresponding author. Email: guilleflores@unc.edu.ar

Email address: guilleflores@unc.edu.ar (Guillermo J. Flores)

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1.2. Hermite type operators and main results

Consider the following positive operator

$$\mathfrak{L} = -\Delta + |x|^2 + \lambda, \qquad \lambda \geq -d.$$

If $\lambda = 0$, this is the well-known *Hermite operator* $\mathfrak{L}_{\mathscr{H}} = -\Delta + |x|^2$, for this reason it can be said that $\mathfrak{L}$ is a Hermite-type operator. If $\lambda = -d$, $\mathfrak{L}$ can be transformed into the *Ornstein-Uhlenbeck operator* $\mathfrak{L}_{\mathscr{O}\mathscr{U}} = -\Delta + 2x \cdot \nabla$, via the formula

$$e^{|x|^2/2}\mathfrak{L}\left(e^{-|x|^2/2}u\right) = -\Delta u + 2x \cdot \nabla u + (\lambda + d)u.$$

Thus, it is usual to say that a function u is $\mathfrak{L}$-harmonic on the upper half-space if it satisfies

$$u_{tt} = \mathfrak{L}u \qquad \text{on } \mathbb{R}^{1+d}_{+}.$$

It is stated in this manner a Calderón theorem for $\mathfrak{L}$-harmonic functions.

Theorem 1.1. *If u is an $\mathfrak{L}$-harmonic function on $\mathbb{R}^{1+d}_{+}$ and is nontangentially bounded at all the points of a set E of positive Lebesgue measure, then u has nontangential limit at almost every point of E.*

This theorem provides a supplementary explanation of the main result, Theorem 1.1, of [4]. Moreover, in these notes the key ideas developed in [4] are analyzed separately and optimized, allowing the following positive operator to be studied.

Consider then the *perturbed Laplacian*

$$L = -\Delta + \alpha, \qquad \alpha > 0.$$

Theorem 1.2. *If u is an L-harmonic function ($u_{tt} = Lu$) on $\mathbb{R}^{1+d}_{+}$ and is nontangentially bounded at all the points of a set E of positive Lebesgue measure, then u has nontangential limit at almost every point of E.*

There are no Calderón theorems beyond those stated in this manuscript, corresponding to [1] and [4], much less generalizations such as the one obtained by L. Carleson.

Throughout these years there has been a significant interest in studying to what extent the fundamental properties of the classical Laplacian can be extended for the operators introduced above, among several others. On the large number of publications about these topics, the following can be cited [5–11] and references therein.

2. Preliminaries

2.1. The Poisson integral associated with $\mathfrak{L}$

A relevant class of $\mathfrak{L}$-harmonic functions on the upper half-space, introduced in [9] (see also [12]), is given by the *Poisson integral* associated with $\mathfrak{L}$ and defined as

$$u(t,x) = \int_{\mathbb{R}^d} P_t^{\mathfrak{L}}(x,y) f(y)\, dy, \tag{2.1}$$

where $P_t^{\mathfrak{L}}$ denotes the $\mathfrak{L}$-*Poisson kernel*

$$P_t^{\mathfrak{L}}(x,y) = \frac{t}{(4\pi)^{(1+d)/2}} \int_0^1 \frac{e^{-\frac{t^2}{2\ln\frac{1+s}{1-s}}}\, e^{-\frac{1}{4}\left(\frac{|x-y|^2}{s} + s|x+y|^2\right)}}{s^{\frac{d}{2}}\left(\frac{1}{2}\ln\frac{1+s}{1-s}\right)^{\frac{3}{2}}} \frac{(1-s)^{\frac{\lambda+d}{2}-1}}{(1+s)^{\frac{\lambda-d}{2}+1}}\, ds,$$

and f is a Lebesgue measurable function belonging to the space

$$\mathscr{L}^1(\phi_\lambda) = \left\{ f : \int_{\mathbb{R}^d} |f(y)| \phi_\lambda(y)\, dy < \infty \right\}$$

for

$$\phi_\lambda(y)=\begin{cases}\dfrac{e^{-|y|^2/2}}{(1+|y|)^{(\lambda+d)/2}[\ln(e+|y|)]^{3/2}} & \text{if} \quad \lambda>-d,\\[2ex] \dfrac{e^{-|y|^2/2}}{[\ln(e+|y|)]^{1/2}} & \text{if} \quad \lambda=-d.\end{cases}$$

As stated in §7.2. "*Non-tangential convergence*" of [9], the $\mathfrak{L}$-harmonic function u has nontangential limit $f(x)$ at almost every $x\in\mathbb{R}^d$.

It is natural to use the notation $e^{-t\sqrt{\mathfrak{L}}}f$ for the above Poisson integral since the *Poisson semigroup* associated with $\mathfrak{L}$, denoted by $\{e^{-t\sqrt{\mathfrak{L}}}\}_{t>0}$ and defined via Bochner's subordination formula

$$e^{-t\sqrt{\mathfrak{L}}}f(x)=\frac{t}{\sqrt{4\pi}}\int_0^\infty \frac{e^{\frac{-t^2}{4s}}}{s^{3/2}}e^{-s\mathfrak{L}}f(x)\,ds, \qquad t>0,$$

see for instance Chapter II of [13], takes the explicit integral representation (2.2). In the last expression, $\{e^{-s\mathfrak{L}}\}_{t>0}$ denotes the *heat-diffusion semigroup* generated by $\mathfrak{L}$ given by

$$e^{-s\mathfrak{L}}f(x)=\int_{\mathbb{R}^d}h_s^{\mathfrak{L}}(x,y)f(y)\,dy,$$

where

$$h_s^{\mathfrak{L}}(x,y)=\frac{e^{-\frac{1}{4}\left(\frac{|x-y|^2}{s}+s|x+y|^2\right)}}{(4\pi s)^{\frac{d}{2}}}\frac{(1-s)^{\frac{\lambda+d}{2}}}{(1+s)^{\frac{\lambda-d}{2}}}$$

is the $\mathfrak{L}$-*heat kernel* (see e.g. [10]). In addition, $u(t,x)=e^{-t\mathfrak{L}}f(x)$ is a well-defined (smooth) function satisfying the following $\mathfrak{L}$-*heat equation*

$$u_t=-\mathfrak{L}u \qquad \text{on } \mathbb{R}_+^{1+d}$$

and has nontangential limit $f(x)$ at almost every $x\in\mathbb{R}^d$.

2.2. *The Poisson integral associated with L*

Similarly, an L-harmonic function on the upper half-space is given by the *Poisson integral* associated with L and defined as

$$u(t,x)=\int_{\mathbb{R}^d}P_t^L(x,y)f(y)\,dy, \tag{2.2}$$

where P_t^L denotes the *L-Poisson kernel*

$$P_t^L(x,y)=\frac{t}{(4\pi)^{(1+d)/2}}\int_0^\infty e^{-s\alpha}\frac{e^{-(t^2+|x-y|^2)/4s}}{s^{(d+3)/2}}\,ds,$$

and f is a Lebesgue measurable function belonging to the space

$$\mathscr{L}^1(\phi)=\left\{f:\int_{\mathbb{R}^d}|f(y)|\phi(y)\,dy<\infty\right\} \qquad \text{for} \qquad \phi(y)=\frac{e^{-\sqrt{\alpha(1+|y|^2)}}}{(1+|y|)^{1+d/2}}.$$

Again, as stated in §7.2. of [9], the L-harmonic function u has nontangential limit $f(x)$ at almost every $x\in\mathbb{R}^d$.

2.3. Pointwise inequalities for the $\mathfrak{L}$-Poisson and L-Poisson kernels

One of the key properties that allow studying several results of Potential Theory for Poisson integrals associated to $\mathfrak{L}$ and L are the following pointwise comparisons between the $\mathfrak{L}$-Poisson and L-Poisson kernels with the usual *Poisson kernel* denoted by

$$p_t(x) = C_d \frac{t}{(t^2+|x|^2)^{(d+1)/2}}, \qquad \text{on } \mathbb{R}^{1+d}_+,$$

where C_d is a positive constant such that $\int_{\mathbb{R}^d} p_1(x)\,dx = 1$. More generally, $C_{\rho_1,\rho_2\ldots}$ shall consistently denote positive constants depending mainly on some parameters $\rho_1, \rho_2, \ldots$, not necessarily the same constants at each occurrence.

Lemma 2.1 (§6.1, Lemma 4.1 and Lemma 4.2 [9]). *For every $(t,x) \in \mathbb{R}^{1+d}_+$ there exist $C_{t,x}$ and $\widetilde{C}_{t,x}$ positive numbers depending mainly on t and x such that*

$$C_{t,x}\,\phi_\lambda(y) \le P^{\mathfrak{L}}_t(x,y) \le \widetilde{C}_{t,x}\,\phi_\lambda(y) \qquad \forall\, y \in \mathbb{R}^d.$$

Also, there exist positive constants γ and C such that

$$P^{\mathfrak{L}}_t(x,y) \le C(1+|x|)^d e^{|x|^2/2}\left[p_t(x-y)e^{-|y|^2/2}\chi_U(y) + t\,\phi_\lambda(y)\right] \qquad \forall\, t>0, x,y \in \mathbb{R}^d,$$

where $U = \{y \in \mathbb{R}^n : |y| \le \gamma \max\{|x|,1\}\}$.

Lemma 2.2 (§5.2, Lemma 5.1 and Lemma 5.2 [9]). *For every $(t,x) \in \mathbb{R}^{1+d}_+$ there exist $C_{t,x}$ and $\widetilde{C}_{t,x}$ positive numbers depending mainly on t and x such that*

$$C_{t,x}\,\phi(y) \le P^{L}_t(x,y) \le \widetilde{C}_{t,x}\,\phi(y) \qquad \forall\, y \in \mathbb{R}^d.$$

Also, there exists a positive constant C such that

$$P^{L}_t(x,y) \le C_\alpha (1+|x|)^{d/2} e^{\sqrt{\alpha}|x|}\left[\max\{t,1\}^{d/2}\frac{p_t(x-y)}{e^{\sqrt{\alpha(1+|y|^2)}}}\chi_U(y) + t\phi(y)\right] \quad \forall\, t>0, x,y \in \mathbb{R}^d,$$

where $U = \{y \in \mathbb{R}^n : |y| \le 2\max\{|x|,1\}\}$.

3. Complementary results

3.1. The $\mathfrak{L}$-Poisson integral of a measure

Let ν be a locally finite complex Borel measure on $\mathbb{R}^d$, that is, ν is finite on compact sets and $\nu = \nu_1 + i\nu_2$, where ν_1 and ν_2 are signed Borel measures. Denote below by $|\nu|$ the *total variation measure* associated with ν.

The pointwise inequalities introduced at the end of the previous section allow to prove without difficulty the assertion below.

Proposition 3.1. *Let ν be a locally finite complex Borel measure. Then, the following conditions are equivalent:*

(i) $\int_{\mathbb{R}^d} P^{\mathfrak{L}}_t(x,y)\,d|\nu|(y) < \infty$, *for all* $(x,t) \in \mathbb{R}^{1+d}_+$.

(ii) $\int_{\mathbb{R}^d} P^{\mathfrak{L}}_{t_0}(x_0,y)\,d|\nu|(y) < \infty$, *for some* $(t_0,x_0) \in \mathbb{R}^{1+d}_+$.

(iii) $\int_{\mathbb{R}^d} \phi_\lambda(y)\,d|\nu|(y) < \infty$.

Furthermore, if ν satisfies the above conditions $(i)-(iii)$, then

$$v(t,x) = \int_{\mathbb{R}^d} P^{\mathfrak{L}}_t(x,y)\,d\nu(y) \in C^\infty\big(\mathbb{R}^{1+d}_+\big).$$

Next, it makes sense to consider the *Poisson integral* associated with $\mathfrak{L}$ of a locally finite complex Borel measure ν defined as

$$e^{-t\sqrt{\mathfrak{L}}}\nu(x) = \int_{\mathbb{R}^d} P_t^{\mathfrak{L}}(x,y)\,d\nu(y), \qquad \text{on } \mathbb{R}_+^{1+d}.$$

Therefore,

$$e^{-t\sqrt{\mathfrak{L}}}|\nu|(x) \text{ is finite for some } (t,x)\in\mathbb{R}_+^{1+d} \quad \text{iff} \quad \int_{\mathbb{R}^d}\phi_\lambda(y)\,d|\nu|(y)<\infty.$$

Thus, as expected, the nontangential limit for $e^{-t\sqrt{\mathfrak{L}}}\nu(x)$ exists a.e.

Theorem 3.2. *If ν is a locally finite complex Borel measure such that*

$$\int_{\mathbb{R}^d}\phi_\lambda(y)\,d|\nu|(y)<\infty,$$

then $u(t,x)=e^{-t\sqrt{\mathfrak{L}}}\nu(x)$ is an $\mathfrak{L}$-harmonic function on $\mathbb{R}_+^{1+d}$ and has nontangential limit $f(x_0)$ at almost every point x_0 of $\mathbb{R}^d$, where f is the Radon-Nikodyn derivative of ν with respect to the Lebesgue measure.

Only the nontangential convergence of last theorem will be proved since the remaining properties are obtained from direct calculations and the previous proposition.

Lemma 3.3. *Let $x_0\in\mathbb{R}^d$ and let n be a natural number such that $|x_0|+2<n$. If ν is a locally finite complex Borel measure such that*

$$\int_{\mathbb{R}^d}\phi_\lambda(y)\,d|\nu|(y)<\infty,$$

then

$$\lim_{(t,x)\to(0,x_0)}\int_{B^c(0,n\gamma)} P_t^{\mathfrak{L}}(x,y)\,d\nu(y)=0,$$

where γ is as in Lemma 2.1 and (t,x) tends to $(0,x_0)$ within $\mathbb{R}_+^{1+d}$.

As usual, $B^c(x,r)$ denotes the complement of the ball centered at x of radius r.

Proof. Since x is tending to x_0, it can be assumed that $|x|+2<n$. Now, applying Lemma 2.1,

$$\left|\int_{B^c(0,n\gamma)} P_t^{\mathfrak{L}}(x,y)\,d\nu(y)\right| \le C_{n,d}\,t\int_{\mathbb{R}^d}\phi_\lambda(y)\,d|\nu|(y)$$

and the proof follows immediately. □

Hence one may focus on

$$\int_{B(0,\gamma n)} P_t^{\mathfrak{L}}(x,y)\,d\nu(y),$$

for $|x|+2<n$. In fact, denote by m the Lebesgue measure. The Lebesgue-Radon-Nikodym decomposition for ν, implies that there exist locally finite complex Borel measures ν_0 and $f\,dm$ such that f is locally Lebesgue integrable, ν_0 and m are mutually singular, $d\nu=f\,dm+d\nu_0$,

$$\lim_{r\to0^+}\frac{\nu\big(B(x,r)\big)}{m\big(B(x,r)\big)}=f(x) \quad \text{and} \quad \lim_{r\to0^+}\frac{|\nu_0|\big(B(x,r)\big)}{m\big(B(x,r)\big)}=0 \quad \text{for almost every } x\in\mathbb{R}^d.$$

And so,

$$\int_{B(0,\gamma n)} P_t^{\mathfrak{L}}(x,y)\,d\nu(y)=\int_{B(0,\gamma n)} P_t^{\mathfrak{L}}(x,y)\,f(y)\,dy+\int_{B(0,\gamma n)} P_t^{\mathfrak{L}}(x,y)\,d\nu_0(y).$$

Lemma 3.4. *Let ν_0 be a locally finite complex Borel measure such that*

$$\int_{\mathbb{R}^d} \phi_\lambda(y)\, d|\nu_0|(y) < \infty \qquad \text{and} \qquad \lim_{r\to 0^+} \frac{|\nu_0|\big(B(x,r)\big)}{m\big(B(x,r)\big)} = 0 \quad \text{a.e. } x \in \mathbb{R}^d.$$

Then, the nontangential limit

$$\lim_{(t,x)\to(0,x_0)} \int_{\mathbb{R}^d} P_t^{\mathfrak{L}}(x,y)\, d\nu_0(y) = 0 \quad \text{a.e. } x_0 \in \mathbb{R}^d.$$

Proof. By Lemma 3.3 it can be assumed that ν_0 has finite total variation, $\|\nu_0\| < \infty$. Also, note that if $x \in \Gamma_\rho(x_0)$ then $p_t(x-y) \le C_\rho\, p_t(x_0-y)$. Thus, by the second inequality of Lemma 2.1

$$\int_{\mathbb{R}^d} P_t^{\mathfrak{L}}(x,y)\, d|\nu_0|(y) \le C_{x_0} \int_{\mathbb{R}^d} p_t(x_0-y)\, d|\nu_0|(y) + C_{x_0} t\, \phi_\lambda(y)\, d|\nu_0|(y),$$

and the second term on the right-side vanishes as $t \to 0^+$. Then, except for a multiplicative constant

$$\begin{aligned}\int_{\mathbb{R}^d} p_t(x-y)\, d|\nu_0|(y) &\le \int_{B(x_0,t)} p_t^\sigma(x_0-y)\, d|\nu_0|(y) + \sum_{j=1}^\infty \int_{A(x_0,2^{j-1}t,2^jt)} p_t^\sigma(x_0-y)\, d|\nu_0|(y)\\ &\le \sum_{j=0}^\infty \frac{1}{2^j} \frac{|\nu_0|\big(B(x_0,2^jt)\big)}{m\big(B(x_0,2^jt)\big)},\end{aligned}$$

and last series vanishes as $t \to 0^+$ since it is convergent and

$$\lim_{t\to 0^+} \sum_{j=0}^n \frac{1}{2^j} \frac{|\nu_0|\big(B(x_0,2^jt)\big)}{m\big(B(x_0,2^jt)\big)} = 0, \qquad \text{for every } n \in \mathbb{N}.$$

□

Proof of Theorem 3.2. For each point x_0, let $n \in \mathbb{N}$ such that $|x_0| + 2 < n$. So, write

$$\begin{aligned}\int_{\mathbb{R}^d} P_t^{\mathfrak{L}}(x,y)\, d\nu(y) &= \int_{B(0,n\gamma)} P_t^{\mathfrak{L}}(x,y)\, d\nu(y) + \int_{B^c(0,n\gamma)} P_t^{\mathfrak{L}}(x,y)\, d\nu(y)\\ &= \int_{B(0,\gamma n)} P_t^{\mathfrak{L}}(x,y)\, f(y)\, dy + \int_{B(0,\gamma n)} P_t^{\mathfrak{L}}(x,y)\, d\nu_0(y)\\ &\quad + \int_{B^c(0,n\gamma)} P_t^{\mathfrak{L}}(x,y)\, d\nu(y),\end{aligned}$$

where γ is the constant from Lemma 2.1 and $d\nu = f dm + d\nu_0$ is the Lebesgue-Radon-Nikodym decomposition for ν mentioned above. The last two summands on the right side vanish as (t,x) tends nontangentially to $(0,x_0)$ (a.e. x_0) by Lemmas 3.3 and 3.4. Also, since

$$\int_{\mathbb{R}^d} |f(y)|\, \chi_{B(0,n\gamma)}(y)\, \phi_\lambda(y)\, dy < \infty,$$

Theorem 1.1 (see also section 7.2) in [9] can be applied to $f(y)\,\chi_{B(0,n\gamma)}(y)$ obtaining that the nontangential limit

$$\lim_{(t,x)\to(0,x_0)} \int_{B(0,\gamma n)} P_t^{\mathfrak{L}}(x,y)\, f(y)\, dy = f(x_0) \qquad \text{a.e. } x_0,$$

as desired to prove. □

3.2. The L-Poisson integral of a measure

The same ideas apply to the L-Poisson integral of ν considering Lemma 2.2 obtaining the following.

Proposition 3.5. *Let ν be a locally finite complex Borel measure. Then, the following conditions are equivalent:*

(i) $\int_{\mathbb{R}^d} P_t^L(x,y)\, d|\nu|(y) < \infty$, *for all* $(x,t) \in \mathbb{R}_+^{1+d}$.

(ii) $\int_{\mathbb{R}^d} P_{t_0}^L(x_0,y)\, d|\nu|(y) < \infty$, *for some* $(t_0,x_0) \in \mathbb{R}_+^{1+d}$.

(iii) $\int_{\mathbb{R}^d} \phi(y)\, d|\nu|(y) < \infty$.

Furthermore, if ν satisfies the above conditions $(i)-(iii)$, then

$$v(t,x) = \int_{\mathbb{R}^d} P_t^L(x,y)\, d\nu(y) \in C^\infty\left(\mathbb{R}_+^{1+d}\right).$$

Lemma 3.6. *Let $x_0 \in \mathbb{R}^d$ and let n be a natural number such that $|x_0| + 2 < n$. If ν is a locally finite complex Borel measure such that*

$$\int_{\mathbb{R}^d} \phi(y)\, d|\nu|(y) < \infty,$$

then

$$\lim_{(t,x)\to(0,x_0)} \int_{B^c(0,2n)} P_t^L(x,y)\, d\nu(y) = 0,$$

where (t,x) tends to $(0,x_0)$ within $\mathbb{R}_+^{1+d}$.

Lemma 3.7. *Let ν_0 be a locally finite complex Borel measure such that*

$$\int_{\mathbb{R}^d} \phi(y)\, d|\nu_0|(y) < \infty \qquad \text{and} \qquad \lim_{r\to 0^+} \frac{|\nu_0|\big(B(x,r)\big)}{m\big(B(x,r)\big)} = 0 \quad \text{a.e. } x \in \mathbb{R}^d.$$

Then, the nontangential limit

$$\lim_{(t,x)\to(0,x_0)} \int_{\mathbb{R}^d} P_t^L(x,y)\, d\nu_0(y) = 0 \quad \text{a.e. } x_0 \in \mathbb{R}^d.$$

Theorem 3.8. *If ν is a locally finite complex Borel measure such that*

$$\int_{\mathbb{R}^d} \phi(y)\, d|\nu|(y) < \infty,$$

then $u(t,x) = e^{-t\sqrt{L}}\nu(x)$ is an L-harmonic function on $\mathbb{R}_+^{1+d}$ and has nontangential limit $f(x_0)$ at almost every point x_0 of $\mathbb{R}^d$, where f is the Radon-Nikodyn derivative of ν with respect to the Lebesgue measure.

4. Proof of Theorems 1.1 and 1.2

4.1. Reductions and notation

Let E be a Lebesgue measurable subset of $\mathbb{R}^d$ with positive measure, and let u be an $\mathfrak{L}$-harmonic function on $\mathbb{R}_+^{1+d}$ that is nontangentially bounded at all the points of E.

One can assume without loss of generality that E is contained in a cube Q centered at some point y_0 with side length 1, since it suffices to the proof for each intersection of E with a member of a mesh of cubes of side 1.

Next, for every positive rational number ρ_j and every natural k, consider

$$F_{\rho_j,k} = \left\{x_0 \in E : |u(t,x)| \leq k,\, (t,x) \in \Gamma_{\rho_j}(x_0)\right\},$$

so that u is uniformly bounded in each $(0,\infty) \times F_{\rho_j,k}$ and E can be written as the countable union of all $F_{\rho_j,k}$. Thus, proving the nontangential convergence of u almost every point of each set $F_{\rho_j,k}$, the proof of the theorem will be completed. So for simplicity, fix an aperture ρ_j and a bound k, and set $F_\rho = F_{\rho_j,k}$.

Now, denote the union of cones with aperture ρ, height 2 and vertices in F_ρ by

$$\mathscr{A} = \bigcup_{x_0 \in F_\rho} \Gamma_\rho(x_0) \cap \{(x,t) : 0 < t < 2\}.$$

One more reduction is that $|u| \leq 1$ on $\mathscr{A}$ can be assumed by dividing by k. Set also

$$\mathscr{B} = \bigcup_{x_0 \in F_\rho} \Gamma_\rho(x_0) \cap \{(t,x) : 0 < t < 1\} \qquad \text{and} \qquad \mathscr{B}_j = \mathscr{A} - \tfrac{1}{j}\, e_{1+d}, \quad j \in \mathbb{N},$$

where $e_{1+d} = (0,\ldots,0,1)$ is the canonical vector in $\mathbb{R}^{1+d}$. Finally, one last notation is

$$\mathscr{G}_j = \mathscr{B}_j \cap \{t = 0\} \hookrightarrow \mathbb{R}^d,$$

where $\{t = 0\}$ denotes the set $\{(0,x) : x \in \mathbb{R}^d\}$. So note that

$$\mathscr{B} \subset \mathscr{A}, \qquad \mathscr{B} \subset \mathscr{B}_j \qquad \text{and} \qquad F_\rho \subset \mathscr{G}_j, \qquad \text{for all } j \in \mathbb{N}.$$

4.2. A special $\mathfrak{L}$-harmonic function

As usual, denote the boundary of $\mathscr{B}$ by $\partial\mathscr{B}$.

Lemma 4.1. *Let M_0 be a positive constant. There exists a nonnegative $\mathfrak{L}$-harmonic function v such that $v(t,x) \geq M_0$ on $\partial\mathscr{B} \setminus \{t = 0\}$ and v has nontangential limit 0 at almost every point of F_ρ.*

Proof. Denote $\chi = \chi_{\mathbb{R}^d \setminus F_\rho}$, the characteristic function of $\mathbb{R}^d \setminus F_\rho$, and define

$$v(t,x) = M \int_{\mathbb{R}^d} \chi(y)\, P_t^{\mathfrak{L}}(x,y)\, dy \qquad \text{on } \mathbb{R}^{1+d}_+,$$

where M will be a positive constant to be chosen later, so that v is nonnegative. Also, since χ is bounded and so

$$\int_{\mathbb{R}^d} \phi_\lambda(y)\, \chi(y)\, dy < \infty,$$

Theorem 3.2 can be applied to the Borel measure $\chi\, dm$ obtaining that v is an $\mathfrak{L}$-harmonic function on the upper half-space and has nontangential limit 0 at almost every point of F_ρ. Thus, it only remains to prove that $v(t,x) \geq M_0$ on $\partial\mathscr{B} \setminus \{t = 0\}$, and this is where the explicit expression of the $\mathfrak{L}$-Poisson kernel

$$P_t^{\mathfrak{L}}(x,y) = \frac{t}{(4\pi)^{(1+d)/2}} \int_0^1 \frac{e^{-\frac{t^2}{2\ln\frac{1+s}{1-s}}}\, e^{-\frac{1}{4}\left(\frac{|x-y|^2}{s} + s|x+y|^2\right)}}{s^{\frac{d}{2}} \left(\frac{1}{2}\ln\frac{1+s}{1-s}\right)^{\frac{3}{2}}} \frac{(1-s)^{\frac{\lambda+d}{2}-1}}{(1+s)^{\frac{\lambda-d}{2}+1}}\, ds,$$

comes into play once again as a key element.

Suppose first that $(t,x) \in \partial\mathscr{B} \setminus \{t = 0\}$ with $t < 1$. Note that the ball centered at x of radius ρt, $B(x,\rho t)$, and F_ρ are disjoint. If this were not the case, there would be a point $x_0 \in B(x,\rho t) \cap F_\rho$, then (t,x) would be an interior point of the cone $\Gamma_\rho(x_0)$. But this contradicts the assumption that $(t,x) \in \partial\mathscr{B}$.

Next, let $0 < \varepsilon < 1/8$. For $0 < s < \varepsilon$, there exist positive constants C_1 and C_2 such that

$$\frac{(1-s)^{\frac{\lambda+d}{2}-1}}{(1+s)^{\frac{\lambda-d}{2}+1}} \geq C_1 \qquad \text{and} \qquad s \leq \ln\frac{1+s}{1-s} \leq C_2\, s.$$

Also for $0<s<\varepsilon$, and since F_ρ is a subset of the cube Q centered at y_0 with side length 1, then there exists a positive C_3 such that

$$\sqrt{s}\,|x+y|<|x|+|y|<2|x|+\rho<C_3(|y_0|+\rho) \qquad \text{for all } y\in B(x,\rho t),$$

and thus, $e^{-\frac{s}{4}|x+y|^2}\geq C_{y_0,\rho}$ for all $y\in B(x,\alpha t)$. Therefore for all $y\in B(x,\rho t)$,

$$\begin{aligned}
P_t^{\mathfrak{L}}(x,y) &\geq C_{y_0,\rho}\, t\int_0^\varepsilon \frac{e^{-\frac{t^2}{s}}\, e^{-\frac{1}{4}\left(\frac{|x-y|}{s}\right)^2}}{s^{\frac{d}{2}}\, s^{\frac{3}{2}}}\,ds\\
&\geq C_{y_0,\rho}\, t\int_0^\varepsilon \frac{e^{-\frac{t^2+|x-y|^2}{s}}}{s^{\frac{d+1}{2}}}\,\frac{ds}{s}\\
\left(\text{change of variables } \tilde{s}=\frac{t^2+|x-y|^2}{s}\right)\qquad &= C_{y_0,\rho}\, t\int_{\varepsilon^{-1}(t^2+|x-y|^2)}^\infty \frac{e^{-\tilde{s}}\,\tilde{s}^{\frac{d+1}{2}}}{(t^2+|x-y|^2)^{\frac{d+1}{2}}}\,\frac{d\tilde{s}}{\tilde{s}}\\
&\geq C_{y_0,\rho}\, p_t(x-y)\int_{\varepsilon^{-1}(1+\rho^2)}^\infty e^{-\tilde{s}}\,\tilde{s}^{\frac{d+1}{2}}\,d\tilde{s}\\
&= C_{y_0,\rho,\varepsilon}\, p_t(x-y). \qquad (4.1)
\end{aligned}$$

Then

$$v(t,x)\geq C_{y_0,\rho,\varepsilon} M\int_{B(x,\rho t)} p_t(x-y)\,dy = C_{y_0,\rho,\varepsilon} M\int_0^\rho \frac{s^{n-1}}{(1+s^2)^{(n+1)/2}}\,ds. \tag{4.2}$$

On the other hand, for $t=1$, there exists $(1,x_1)\in\partial\mathscr{B}$ such that $|x_1|\leq|y_0|+\rho$ and $B(x_1,\rho)\cap F_\rho=\emptyset$. Therefore, for $(1,x)\in\partial\mathscr{B}$ and $y\in B(x_1,\rho)$, clearly

$$|x+y|\leq|x|+|y-x_1|+|x_1|\leq C_{y_0,\rho}.$$

Proceeding as in the previous part,

$$v(1,x)=M\int_{\mathbb{R}^d}\chi(y)\,P_1^{\mathfrak{L}}(x,y)\,dy\geq C_{y_0,\rho,\varepsilon} M\int_0^\rho \frac{s^{n-1}}{(1+s^2)^{(n+1)/2}}\,ds. \tag{4.3}$$

Both expressions (4.2) and (4.3) can obviously be made to exceed M_0 by choosing M appropriately. □

4.3. Proof of Theorem 1.1

Proof. After the introduced notation, for every natural j consider the translations of u in the t-variable

$$u_j(t,x)=u\big(t+\tfrac{1}{j},x\big),$$

and denote $\chi_j(x)=\chi_{\mathscr{G}_j}(x)$, where $\chi_{\mathscr{G}_j}$ is the characteristic function of $\mathscr{G}_j$. Now, consider the $\mathfrak{L}$-Poisson integral of the function $f_j(y)=\chi_j(y)\,u_j(0,y)$,

$$\varphi_j(t,x)=\int_{\mathbb{R}^d} f_j(y)\,P_t^{\mathfrak{L}}(x,y)\,dy, \qquad \text{on } \mathbb{R}_+^{1+d},$$

which is well-defined since

$$\int_{\mathbb{R}^d}\phi_\lambda(y)\,|f_j(y)|dy\leq\int_{\mathscr{G}_j} e^{-|y|^2/2}\,dy<\infty,$$

and has nontangential limit $f_j(x_0)$ a.e. $x_0\in\mathbb{R}^d$ by Theorem 3.2.

Next define the $\mathfrak{L}$-harmonic functions

$$\psi_j(t,x) = u_j(t,x) - \varphi_j(t,x) \qquad \text{on } \mathbb{R}^{1+d}_+.$$

Note now that $\|f_j\|_1$ is uniformly bounded, in fact $\|f_j\|_1 \le C(1+2\rho)^d$ for all $j \in \mathbb{N}$. Then by Helly's theorem, there exists a subsequence $\{f_{j_k}\}$ and a finite Borel measure ν (depending mainly on y_0 and ρ) such that

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} f_{j_k}\, g = \int_{\mathbb{R}^d} g\, d\nu, \qquad \text{for all } g \in C_0(\mathbb{R}^d).$$

In particular,

$$\int_{\mathbb{R}^d} \phi_\lambda(y)\, d|\nu|(y) \le |\nu|(\mathbb{R}^d) < \infty,$$

and since $P_t^{\mathfrak{L}}(x,\cdot) \in C_0(\mathbb{R}^d)$, it follows that

$$\lim_{k\to\infty} \underbrace{\int_{\mathbb{R}^d} f_{j_k}(y)\, P_t^{\mathfrak{L}}(x,y)\, dy}_{\varphi_{j_k}(t,x)} = \int_{\mathbb{R}^d} P_t^{\mathfrak{L}}(x,y)\, d\nu(y) \qquad \text{on } \mathbb{R}^{1+d}_+.$$

Hence, the function

$$\varphi(t,x) = \int_{\mathbb{R}^d} P_t^{\mathfrak{L}}(x,y)\, d\nu(y) \qquad \text{on } \mathbb{R}^{1+d}_+$$

is $\mathfrak{L}$-harmonic and has nontangential limit $f(x_0)$ a.e. $x_0 \in \mathbb{R}^d$, where f is the Radon-Nikodyn derivative of ν with respect to the Lebesgue measure by Theorem 3.2. Moreover,

$$\lim_{k\to\infty} \psi_{j_k}(t,x) = u(t,x) - \varphi(t,x) \qquad \text{on } \mathbb{R}^{1+d}_+, \tag{4.4}$$

so set $\psi(t,x) = u(t,x) - \varphi(t,x)$. Then, $u = \varphi + \psi$ and the proof will be completed by showing that ψ has nontangential limit 0 for almost every point in F_ρ.

Note first that

$$\lim_{(t,x)\to(0,x_0)} \psi_{j_k}(t,x) \;=\; u_{j_k}(0,x_0) - f_{j_k}(x_0) \;=\; 0 \qquad \text{a.e. } x_0$$

where (t,x) tends $(0,x_0)$ within $\mathscr{B}$, since $F_\rho \subset \mathscr{G}_j$ for every $j \in \mathbb{N}$. But moreover, note that if $x_0 \in \mathscr{G}_j$, then x_0 is a Lebesgue point of f_j and Theorem 1.1 of [14] can be applied obtaining that

$$\lim_{(t,x)\to(0,x_0)} \psi_{j_k}(t,x) \;=\; u_{j_k}(0,x_0) - f_{j_k}(x_0) \;=\; 0 \qquad \text{for every } x_0 \in \mathscr{G}_{j_k} \tag{4.5}$$

where (t,x) tends $(0,x_0)$ within $\mathscr{B}$.

On the other hand, by Lemma 2.1

$$\begin{aligned} |\varphi_j(t,x)| &\le C(1+|x|)^d e^{|x|^2/2} \int_{\mathbb{R}^d} |f_j(y)| \big[p_t(x-y)e^{-|y|^2/2}\chi_U(y) + t\,\phi_\lambda(y)\big]\, dy \\ &\le C(1+|x|)^n e^{|x|^2/2}(1+t) \qquad \text{on } \mathbb{R}^{1+d}_+. \end{aligned}$$

Therefore,

$$|\psi_{j_k}(t,x)| \le |u_{j_k}(t,x)| + |\varphi_{j_k}(t,x)| \le 1 + Ce^{|x|^2/2}(1+t)(1+|x|)^d \le M_{y_0,\rho} \tag{4.6}$$

on $\mathscr{B}$, and thus on $\partial\mathscr{B}\setminus\{t=0\}$, where it has been previously established that $M_{y_0,\rho}$ is a constant depending mainly on y_0 and ρ. Applying Lemma 4.1, there is a nonnegative $\mathfrak{L}$-harmonic function v such that $v(t,x)\geq M_{y_0,\rho}$ on $\partial\mathscr{B}\setminus\{t=0\}$ and v has nontangential limit 0 at almost every point of F_ρ. It follows from (4.6) and (4.5) that, for all j_k

$$v(t,x)\pm\psi_{j_k}(t,x)\geq 0 \qquad \text{on } \partial\mathscr{B}\setminus\{t=0\}$$

and

$$\liminf_{(t,x)\to(0,x_0)} v(t,x)\pm\psi_{j_k}(t,x)\geq \liminf_{(t,x)\to(0,x_0)} v(t,x)\geq 0,$$

where (t,x) tends $(0,x_0)$ within $\mathscr{B}$. Next, a standard argument based on the *Minimum Principle* (see for instance §18.2 and a corollary of *The Hopf Maximum principle*, Theorem 18.5, in [15]) will show that $v(t,x)\pm\psi_{j_k}(t,x)\geq 0$ in $\mathscr{B}$. Assume for contradiction that there exist $\varepsilon>0$ and a point $(t^*,x^*)\in\mathscr{B}$ such that $v(t^*,x^*)+\psi_{j_k}(t^*,x^*)=-2\varepsilon<0$. Since $v+\psi_{j_k}$ is nonnegative and continuos on $\partial\mathscr{B}\setminus\{t=0\}$, the minimum principle implies the existence of a sequence $\{(t_i^*,x_i^*)\}_{i\in\mathbb{N}}$ in $\mathscr{B}$ that converges to $(0,x_0)$ for some $x_0\in F_\rho$ such that $v(t_i^*,x_i^*)+\psi_{j_k}(t_i^*,x_i^*)<-\varepsilon$ for all i. But for i sufficiently large, the number $\psi_{j_k}(t_i^*,x_i^*)$ is very close to 0 by (4.5), so that $v(t_i^*,x_i^*)$ would be negative for such i, which contradicts that v is nonnegative.

Therefore it follows that

$$|\psi_{j_k}(t,x)|\leq v(t,x) \qquad \text{on } \mathscr{B},\text{ for all } j_k,$$

and then

$$|\psi(t,x)|\leq v(t,x) \qquad \text{on } \mathscr{B}.$$

This implies that

$$\lim_{(t,x)\to(0,x_0)} \psi(t,x)\,=\,0 \qquad \text{a.e. } x_0\in F_\rho$$

where (t,x) tends $(0,x_0)$ within $\mathscr{B}$. It will now be proven that ψ has nontangential limit 0 at almost every point of F_ρ.

Note that almost every point of F_ρ is a point of density of F_ρ, that is

$$\lim_{r\to 0^+}\frac{1}{m(B(x_0,r))}\int_{B(x_0,r)}\chi_{F_\rho}(y)\,dy=1 \qquad \text{a.e. } x_0\in F_\rho.$$

simply because χ_{F_ρ} is a locally integrable function. Thus, if x_0 is a point of density of F_ρ and $\Gamma_\beta(x_0)$ is a cone with arbitrary aperture β, then the inequality $|\psi|\leq v$ on $\mathscr{B}$, implies that $|\psi(t,x)|\leq v(t,x)$ holds for (t,x) sufficiently close to $(0,x_0)$ within $\Gamma_\beta(x_0)$. Consequently, ψ has nontangential limit 0 at almost every point of F_ρ since v has it, and the proof is completed. □

4.4. *Proof of Theorem 1.2*

In continuation of the reductions and notation introduced at the beginning of the previous section, it is obtained that.

Lemma 4.2. *Let M_0 be a positive constant. There exists a nonnegative L-harmonic function v such that $v(t,x)\geq M_0$ on $\partial\mathscr{B}\setminus\{t=0\}$ and v has nontangential limit 0 at almost every point of F_ρ.*

The proof of this lemma is similar to the proof of Lemma 4.1 since Theorem 3.8 allows to define the L-harmonic function

$$v(t,x)=M\int_{\mathbb{R}^d}\chi(y)\,P_t^L(x,y)\,dy \qquad \text{on } \mathbb{R}_+^{1+d},$$

where M is a positive constant to be chosen conveniently. At the same time, for $(t,x)\in\partial\mathscr{B}\setminus\{t=0\}$,

$$P_t^L(x,y)\geq C_{y_0,\rho}\,t\int_0^\varepsilon\frac{e^{-\frac{t^2+|x-y|^2}{s}}}{s^{\frac{d+1}{2}}}\frac{ds}{s}\geq C_{y_0,\rho,\varepsilon}\,p_t(x-y),$$

as was shown in (4.1). Thus the function v is the desired one.

Finally, the proof of Theorem 1.2 can be obtained by following the main arguments of Theorem 1.1.

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Use of AI Tools

AI tools were not employed in generating, analyzing, or interpreting the results.

References

[1] A. P. Calderón, *On the Behaviour of Harmonic Functions at the Boundary*, Transactions of the American Mathematical Society, 68(1), 1950, 47–54. DOI: 10.2307/1990537

[2] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 32, 1971. DOI: 10.2307/j.ctt1bpm9w6

[3] L. Carleson, *On the existence of boundary values for harmonic functions in several variables*, Arkiv för Matematik, 4(5), 1962, 393–399. DOI: 10.1007/BF02591620

[4] G. Flores and B. Viviani, *A Calderón theorem for the Poisson semigroups associated with the Ornstein–Uhlenbeck and Hermite operators*, Mathematische Annalen, 386, 2023, 329–342. DOI: 10.1007/s00208-022-02538-7

[5] L. Forzani and W. Urbina, *Poisson–Hermite representation of solutions of the equation* $\frac{\partial^2}{\partial t^2}u(x,t)+\Delta_x u(x,t)-2x\cdot\nabla_x u(x,t)=0$, in: M. Lassonde (ed.), *Approximation, Optimization and Mathematical Economics*, Physica, Heidelberg, 2001, 109–115. DOI: 10.1007/978-3-642-57592-1_9

[6] L. Liu and P. Sjögren, *A characterization of the Gaussian Lipschitz space and sharp estimates for the Ornstein–Uhlenbeck Poisson kernel*, Revista Matemática Iberoamericana, 32, 2016, 1189–1210. DOI: 10.4171/RMI/912

[7] E. Pineda and W. Urbina, *Non tangential convergence for the Ornstein–Uhlenbeck semigroup*, Divulgaciones Matemáticas, 16(1), 2008, 107–124.

[8] P. Sjögren and J. L. Torrea, *On the boundary convergence of solutions to the Hermite–Schrödinger equation*, Colloquium Mathematicum, 118, 2010, 161–174. DOI: 10.4064/cm118-1-8

[9] G. Garrigós, S. Hartzstein, T. Signes, J. L. Torrea, and B. Viviani, *Pointwise convergence to initial data of heat and Laplace equations*, Transactions of the American Mathematical Society, 368, 2016, 6575–6600. DOI: 10.1090/tran/6554

[10] S. Thangavelu, *Lecture Notes on Hermite and Laguerre Expansions*, Princeton University Press, Princeton, NJ, 1993.

[11] W. Urbina-Romero, *Gaussian Harmonic Analysis*, Springer Monographs in Mathematics, Springer, Cham, 2019. DOI: 10.1007/978-3-030-05597-4

[12] G. Garrigós, *A weak 2-weight problem for the Poisson–Hermite semigroup*, Advanced Courses of Mathematical Analysis VI (Málaga, 2014), World Scientific, Hackensack, NJ, 2017, 153–171.

[13] E. M. Stein, *Topics in Harmonic Analysis Related to the Littlewood–Paley Theory*, Annals of Mathematical Studies Princeton University Press, Princeton, NJ, 63, 1970.

[14] G. Flores, G. Garrigós, and B. Viviani, *Lebesgue points of measures and non-tangential convergence of Poisson–Hermite integrals*, Journal of Evolution Equations, 25, 2025, Art. 50, pp. 1–18. DOI: 10.1007/s00028-025-01079-5

[15] M. Chipot, *Elliptic Equations: An Introductory Course*, Birkhäuser, Basel, 2009. DOI: 10.1007/978-3-7643-9982-5