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Review Article

A Summarize of Research on Schur Convexity Related to Hadamard Integral Inequality

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Abstract

In 2000, Croatian mathematicians Elezović and Pečarić investigated the Schur-convexity of the integral mean of a convex function with respect to the upper and lower limits of integration, obtaining an important and pioneering result. Building on this work, numerous scholars at home and abroad have since carried out a series of generalizations and extensions. This paper presents a survey of these developments, with the aim of promoting deeper research on integral inequalities via majorization theory.

Keywords: Integral mean of convex functions, Schur-convexity, majorization, summarize

2020 MSC: 26E60, 26A51, 26D15, 26D20.

1. Introduction

In 1979, A. M. Marshall and I. Olkin co-published "Inequalities: Theory of Majorization and Its Application"[1]. Since then, majorization theory has become an independent discipline of mathematics. 2011, A. M. Marshall, I. Olkin and B.C. Arnold published Inequalities: Theory of Majorization and Its Application (The second edition)[2].

In 1990, Professor Boying Wang's book "Fundamentals of Majorization Inequalities" (in Chinese)[3] was published. In addition to the classical basic theories in Marshall and Olkin's book, this book also contains a number of wonderful original contents of Wang professor.

As Wang Professor puts it, "The majorization inequalities have almost infiltrated into various fields of mathematics, and played wonderful role everywhere, because it can always profoundly describe the intrinsic relationship between many mathematical quantities, thus facilitating the derivation of the required conclusions. It can also easily derive many existing inequalities derived from different methods in a uniform way. It is a powerful means to generalize existing inequalities and discover new inequalities, and the theory and application of the majorization inequalities have a bright future."

In 2019, Huan-nan Shi's English mathematics monograph "Schur Convex Functions and Inequalities" was jointly published by Harbin Institute of Technology Press Co., Ltd. and Germany's Walter de Groot Co., Ltd. (Walter de Gruyter GmbH) in two volumes [4]. This monograph is supported by the National Science and Technology academic works Publishing Fund in 2019. Currently, Huan-nan Shi is reviewing the second edition of this monograph.

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2. Schur convex function and its generalizations

In this article, $\mathbb{R}^n$, $\mathbb{R}^n_+$, $\mathbb{R}^n_{++}$ and $\mathbb{R}^n_{--}$ represents n dimensional real number set, n dimensional non negative real number set, n dimensional positive real number set and n dimensional negative real number set respectively, and $\mathbb{R}^1 = \mathbb{R}, \mathbb{R}^1_+ = \mathbb{R}_+$ and $\mathbb{R}^1_{++} = \mathbb{R}_{++}$.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be convex set. A function φ: $\Omega \to \mathbb{R}$ is said to be a convex function on Ω if

$$c\varphi(\alpha x + (1-\alpha)y) \leq \alpha\varphi(x) + (1-\alpha)\varphi(y) \tag{2.1}$$

holds for all $x, y \in \Omega$, and all $\alpha \in [0,1]$. If $-\varphi$ is convex, φ is said to be concave.

Definition 2.2. [1–3] For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, arrange the components of x in descending order, and record them as $x_{[1]} \geq x_2 \geq \cdots \geq x_{[n]}$. Let $x, y \in \mathbb{R}^n$ satisfy

$(i) \sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}, k = 1, 2, \ldots, n-1,$ $(ii) \sum_{i=1}^n x_i = \sum_{i=1}^n y_i,$

it is said that x is said to be majorized by y and in symbols $x \prec y$.

$x \leq y$ represents $x_i \leq y_i, i = 1, \ldots, n$.

Definition 2.3. [1–3] Let $\Omega \subset \mathbb{R}^n, \varphi : \Omega \to \mathbb{R}$, if $x \prec y \Rightarrow \varphi(x) \leq \varphi(y)$ on Ω, then φ is called the Schur convex function (S-convex function for short) on Ω; If $-\varphi$ is an S-convex function on Ω, then φ is called an S-concave function on Ω.

If $x \leq y \Rightarrow \varphi(x) \leq \varphi(y)$ is on Ω, φ is said to be an increasing function on Ω; If $-\varphi$ is an increasing function on Ω, then φ is called a decreasing function on Ω.

Lemma 2.4. *[1–3] (Schur convex function decision theorem)*

Let $\Omega \subset \mathbb{R}^n$ be a convex set, and has a nonempty interior set Ω° . If $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω°, then φ is $Schur-convex(or\ Schur-concave, resp.)$, if and only if it is symmetric on Ω and

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \geq 0\ (or\ \leq 0, resp.) \tag{2.2}$$

holds for any $x = (x_1, \cdots, x_n) \in \Omega^\circ$.

Chinese scholars have done a lot of pioneering work on the generalization of Schur-convex functions, which have been recognized and applied by foreign scholars. In 2003, Xiaoming Zhang first proposed the definition of Schur geometrically convex function and and the corresponding decision theorem is established.

Definition 2.5. [5, 6] Let $\Omega \subset \mathbb{R}^n_{++}$, $f : \Omega \to \mathbb{R}_+$, for any $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \Omega$, if

$$\ln x := (\ln x_1, \ldots, \ln x_n) \prec (\ln y_1, \ldots, \ln y_n) =: \ln y$$

implies $f(x) \leq f(y)$, then it is called f is the S-geometrically convex function on Ω. If $\ln x \prec \ln y$ has $f(x) \geq f(y)$, then f is called an S-geometrically concave function on Ω.

Proposition 2.6. *Let $\Omega \subset \mathbb{R}^n_+$. Set $\ln\Omega = \{(\ln x_1, \ldots, \ln x_n) : (x_1, \ldots, x_n) \in \Omega\}$. Then $\varphi : \Omega \to \mathbb{R}_+$ is Schur-geometrically convex (or Schur-geometrically concave, resp.) on Ω if and only if $\varphi(e^{x_1}, \ldots, e^{x_n})$ Schur convex (or Schur concave) on $\ln\Omega$.*

Lemma 2.7. *[5, 6] (Schur geometrically convex function decision theorem)*

Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric geometrically convex set with a nonempty interior Ω° and $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable on Ω°. Then φ is a Schur-geometrically convex (or Schur-geometrically concave, resp.) function if and only if φ is symmetric on Ω and

$$(\ln x_1 - \ln x_2)\left(x_1\frac{\partial\varphi(x)}{\partial x_1} - x_2\frac{\partial\varphi(x)}{\partial x_2}\right) \geq 0 \quad (or \leq 0, resp.)$$

holds for any $x = (x_1, \cdots, x_n) \in \Omega^\circ$.

In 2008, Yuming Chu et al first proposed and established the definition and the corresponding decision theorem is established.

Definition 2.8. [6] Let $\Omega \subset \mathbb{R}^n_{++}$ or $\Omega \subset \mathbb{R}^n_{--}$, $\varphi : \Omega \to \mathbb{R}_+$. $\forall x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in \Omega$. If

$$\left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right) \prec \left(\frac{1}{y_1}, \ldots, \frac{1}{y_n}\right) \Rightarrow \varphi(x) \leq \varphi(y),$$

then φ is called a S-harmonically convex function on Ω, and if $-\varphi$ is a S-harmonically convex function on Ω, then φ is called a S-harmonically concave function on Ω.

Proposition 2.9. *Let $\Omega \subset \mathbb{R}^n_{++}$. $\varphi : \Omega \to \mathbb{R}_+$ is an S-harmonically convex function if and only if $\varphi\left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right)$ is a S-convex function on*

$$\frac{1}{\Omega} = \left\{\frac{1}{x} = \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right) \mid x = (x_1, \ldots, x_n) \in \Omega\right\}.$$

Lemma 2.10. *[6] (Determination theorem of Schur harmonically convex function)*

Let $\Omega \subset \mathbb{R}^n_{++}$ or $\Omega \subset \mathbb{R}^n_{--}$ be a symmetric and harmonically convex set with inner points, and let $\varphi : \Omega \to \mathbb{R}$ be a continuously symmetric function which is differentiable on interior Ω°. Then φ is S-harmonically convex (or S-harmonically concave, resp.) on Ω if and only if

$$(x_1 - x_2)\left(x_1^2 \frac{\partial \varphi(x)}{\partial x_1} - x_2^2 \frac{\partial \varphi(x)}{\partial x_2}\right) \geq 0 \quad (or \leq 0,\ resp.), \quad \forall x \in \Omega^\circ. \tag{2.3}$$

Remark 2.11. Note that the definition and determination theorem of Schur harmonically convex function established by Chu Yuming are extended as follows:

(a) Expand $\Omega \subset \mathbb{R}^n_+$ to $\Omega \subset \mathbb{R}^n_+$ or $\Omega \subset \mathbb{R}^n_-$;
(b) $\varphi : \Omega \to \mathbb{R}$ does not have to be a positive function.

As a unified generalization of the concepts of Schur convex function, Schur geometrically convex function and Schur harmonically convex function, Zhenhang Yang defined Schur f convex function and Schur power convex function in 2010, and the corresponding decision theorem is established.

Definition 2.12. [7] Let $f : \mathbb{R}_+ \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} \dfrac{x^m - 1}{m}, & m \neq 0; \\ \ln x, & m = 0. \end{cases}$$

For any $x, y \in \Omega$, if $f(x) \prec f(y)$ implies $\phi(x) \leq \phi(y)$, then say $\phi : \Omega \subset \mathbb{R}^n_+ \to \mathbb{R}$ is the Schur m-power convex function on Ω. If $-\phi$ is a Schur m-power convex function, then ϕ is said to be a Schur m-power concave function on Ω.

In Definition 2.6, if $f(x)$ is defined as $x, \ln x$ and $\frac{1}{x}$ respectively, then Definition 2.6 is transformed into the definition of Schur-convex, Schur-geometrically convex and Schur-harmonically convex functions.

Lemma 2.13. *[7](Decision Theorem of Schur power convex function).*

Let $\Omega \subset \mathbb{R}^n$ be a symmetric convex set with a nonempty interior Ω°, $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω, and is differentiable on Ω°, then φ is Schur m-power convex (or Schur m-power concave, resp.) on Ω if and only if φ is symmetric on Ω and

$$\frac{x_1^m - x_2^m}{m}\left(x_1^{1-m} \frac{\partial \varphi(x)}{\partial x_1} - x_2^{1-m} \frac{\partial \varphi(x)}{\partial x_2}\right) \geq (or\ \leq 0, resp.),\ if\ m \neq 0 \tag{2.4}$$

$$(\ln x_1 - \ln x_2)\left(x_1 \frac{\partial \varphi(x)}{\partial x_1} - x_2 \frac{\partial \varphi(x)}{\partial x_2}\right) \geq 0 \quad (or \leq 0, resp.),\ if\ m = 0 \tag{2.5}$$

holds for any $x = (x_1, \ldots, x_n) \in \Omega^\circ$ *with* $x_1 \neq x_2$.

For $m \neq 0$, *since*

$$\left(\frac{1}{n}\sum_{i=1}^{n} \frac{x_i^m - 1}{m}, \ldots, \frac{1}{n}\sum_{i=1}^{n} \frac{x_i^m - 1}{m}\right) \prec \left(\frac{x_1^m - 1}{m}, \ldots, \frac{x_n^m - 1}{m}\right),$$

this is

$$\left(\frac{\left(\left(\frac{1}{n}\sum_{i=1}^{n} x_i^m\right)^{\frac{1}{m}}\right)^m - 1}{m}, \ldots, \frac{\left(\left(\frac{1}{n}\sum_{i=1}^{n} x_i^m\right)^{\frac{1}{m}}\right)^m - 1}{m}\right) \prec \left(\frac{x_1^m - 1}{m}, \ldots, \frac{x_n^m - 1}{m}\right), \tag{2.6}$$

if φ *is on* Ω *Schur m-power convex (or Schur m-power concave, resp.), then*

$$\varphi\left(M_m(x), \ldots, M_m(x)\right) \leq (or \geq, resp.)\, \varphi\left(x_1, \ldots, x_n\right). \tag{2.7}$$

where $M_m(x) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^m\right)^{\frac{1}{m}}$ *is power mean of* x.

3. Schur convexity of the integral mean of convex functions

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex (or concave, resp.) function defined on the real number interval I, $x, y \in I$, $x < y$. Then the following two-sided inequalities holds.

$$f\left(\frac{x+y}{2}\right) \leq (or \geq, resp.) \frac{1}{y-x}\int_x^y f(t)dt \leq (or \geq, resp.) \frac{f(x)+f(y)}{2}, \tag{3.1}$$

This is the famous Hadamard inequality for convex functions.

There are many ways to prove the Hadamard inequality. Zheng [8] given a majorized proof.

The research on Hadamard inequality has been lasting for a long time, and there are a large number of literatures.

In 2000, Elezovic and Pecaric [9] considered the Schur-convexity of the integral mean of a convex function with respect to its upper and lower limits of integration, and by using Hadamard's inequality they obtained the following result.

Theorem 3.1. *[9] Let I be an interval with nonempty interior on* $\mathbb{R}$ *and* f *be a continuous function on* I. *Then*

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt, & x, y \in I,\ x \neq y \\ f(x), & x = y \end{cases}$$

is Schur-convex (or Schur-concave, resp.) on $I \times I$ *if and if* f *is convex (or concave, resp.) on* I.

About in July and August 2001, I went to the National Library to search for information and saw the concise and beautiful papers of Elezovic and Pecaric. I was very excited at the time because I had been trying to apply majorization theory to some integral inequalities, but there has been no train of thought, this article makes me open the door. Soon after getting this article, I will send a copy to a friend Prof. Feng Qi to share. Feng Qi immediately wrote back and said: This article is too important to me. Soon Feng Qi wrote an article [10].

Theorem 3.1 is an important and groundbreaking result. It provides the first example of a Schur-convex function expressed in integral form and opens the way to studying integral inequalities by means of majorization theory. Inspired by this pioneering work, a large number of scholars at home and abroad have subsequently carried out a series of extensions and generalizations. This paper surveys these developments, with the aim of promoting deeper applications of majorization theory to integral inequalities.

In recent years, this result has attracted the attention of many scholars. Zhang and Chu [11] considered convexity of $F(x,y)$ and obtained the following result.

Theorem 3.2. *[11] Let f be a continuous function on I, then $F(x,y)$ defined is a convex $($or concave, resp.$)$ function on I^2 if and only if f is a convex $($or concave, resp.$)$ function on I.*

Remark 3.3. The symmetric convex $($*or concave*, *resp.*$)$ function on the symmetric convex set must be an Schur-convex (or Schur-concave, respectively) function. Therefore, Theorem 3.2 is the strengthen of Theorem 3.1. In 2003, Wulbert [12] proved sufficiency that if f is a convex function on I, then F is a convex function on I^2.

Long et al. [13] and Sun et al. [14] proved the following theorem in different ways.

Theorem 3.4. *Let f be an increasing $($or decreasing, resp.$)$ continuous convex $($ or concave, resp.$)$ function on $I \subset \mathbb{R}_{++}$. Then $F(x,y)$ in Theorem 2.1 is both a Schur-geometrically convex $($or Schur-geometrically concave, resp.$)$ function and a Schur-harmonically convex $($ or Schur-harmonically concave, resp.$)$ function on $I \times I$.*

In 2005, Qi et al. [15] established the weighted form of Theorem 3.1.

Theorem 3.5. *Let f be a continuous function on I and p be a positive continuous function on I. Then the weighted arithmetic mean of f with weight p*

$$F_p(f;a,b) = \begin{cases} \frac{\int_x^y p(t)f(t)dt}{\int_x^y p(t)dt}, & x \neq y; \\ f(x), & x = y. \end{cases} \tag{3.2}$$

is the Schur-convex $($ or Schur-concave, resp.$)$ function on $I \times I$ if and only if inequality

$$\frac{\int_x^y p(t)f(t)dt}{\int_x^y p(t)dt} \leq \frac{p(x)f(x)+p(y)f(y)}{p(x)+p(y)} \tag{3.3}$$

holds $($ or inverted, resp.$)$ for $(x,y) \in I \times I$.

In 2013, Long et al. [13] further investigated the Schur-geometric convexity and Schur-harmonic convexity of $F_p(x,y)$, and obtained the following results:

Theorem 3.6. *[13] The same condition as the Theorem 3.4, then*

(a) *$F_p(x,y)$ is a Schur-geometrically convex function on $I \times I$ if and only if inequality*

$$\frac{\int_x^y p(t)f(t)dt}{\int_x^y p(t)dt} \leq \frac{xp(x)f(x)+yp(y)f(y)}{xp(x)+yp(y)} \tag{3.4}$$

holds for $(x,y) \in I \times I$;

(b) *$F_p(x,y)$ is a Schur-harmonically convex function on $I \times I$ if and only if the inequality*

$$\frac{\int_x^y p(t)f(t)dt}{\int_x^y p(t)dt} \leq \frac{x^2p(x)f(x)+y^2p(y)f(y)}{x^2p(x)+y^2p(y)} \tag{3.5}$$

holds for $(x,y) \in I \times I$.

Theorem 3.7. *[13] Let p be a positive continuous function on I, f be a differentiable function on I, and satisfy any $x,y \in I$*

$$f^{'}(y) \geq \frac{p(y)}{\int_x^y p(t)dt} \cdot \frac{f(y)-f(x)}{y-x}.$$

Then the following propositions are established:

(a) *If f and p have the same monotonicity on I, then $F_p(x,y)$ is Schur-convex on $I \times I$;*

(b) *if $f(t)$ and $tp(t)$ have the same monotonicity on I, then $F_p(x,y)$ is Schur-geometrically convex on $I \times I$;*

(c) *if $f(t)$ and $t^2p(t)$ have the same monotonicity on I, then $F_p(x,y)$ is Schur-harmonically convex on $I \times I$.*

In 2011, Culjak [16] of Croatia extended the above results to the case of a generalized integral quasiarithmetic mean:

Theorem 3.8. *Let f be a Lebesgue integrable function with the range J on the interval $I \subset \mathbb{R}$, and k be a continuous and strictly monotonous real function on J, then for the generalized integral quasiarithmetic mean of the function f,*

$$M_k(f;x,y) = \begin{cases} k^{-1}\left(\frac{1}{y-x}\int_x^y (k\circ f)(t)dt\right), & x \neq y; \\ f(a), & x = y, \end{cases} \tag{3.6}$$

the following propositions hold true:

- (a) *If $k\circ f$ is convex on I and k increases on J or $k\circ f$ is concave on I and k decreases on J, then $M_k(f;x,y)$ is Schur-convex on I^2;*
- (b) *If $k\circ f$ is convex on I and k decreases on J or $k\circ f$ is concave on I and k increases on J, then $M_k(f;x,y)$ is Schur-concave on I^2.*

In 2010, Yuming Chu et al. [17] proved the following theorem.

Theorem 3.9. *Let I be an open interval and $f : I \to \mathbb{R}$ be a continuous function, if*

$$F_f(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt - f\left(\frac{x+y}{2}\right), & x,y\in I, x\neq y; \\ 0, & x = y. \end{cases} \tag{3.7}$$

$$G_f(x,y) = \begin{cases} \frac{f(x)+f(y)}{2} - \frac{1}{y-x}\int_x^y f(t)dt, & x,y\in I, x\neq y; \\ 0, & x = y, \end{cases} \tag{3.8}$$

then $F_f(x,y)$ and $G_f(x,y)$ are Schur-convex (or Schur-concave, resp.) functions on I^2 if and only if f is a convex (or concave, resp.) function on I.

In 2016, Shuhong Wang [18] extended Theorem 3.9 as follows.

Theorem 3.10. *Let I be a non-empty interval on $\mathbb{R}$. The function $S : I^2 \to \mathbb{R}$ is defined as*

$$S(x,y) = \begin{cases} \lambda[f(x)+f(y)] + (1-2\lambda) f\left(\frac{x+y}{2}\right) - \frac{1}{y-x}\int_x^y f(t)dt, & x \neq y; \\ 0, & x = y. \end{cases} \tag{3.9}$$

where $(x,y) \in I^2, \lambda \geq 0$.

Theorem 3.11. *Let $I \subset \mathbb{R}$ be an open interval, and $f \to \mathbb{R}$ be a second-order differentiable mapping, such that f'' is integrable. If $\lambda \geq \frac{1}{4}$ and f is convex (or concave, resp.) on I, then the function $S(x,y)$ is Schur-convex (or Schur-concave, resp.) on I^2.*

In 2011, V. Culjak etal. [19] and Franjic etal. [20] proved Theorem 8 for $\lambda = \frac{1}{2}$ and $\lambda = \frac{1}{3}$ respectively.

Since 2005, Huan-nan Shi et al. have studied some Schur-convexity involving Hadamard-type inequalities.

Let f be continuous on I, for $t \in [0,1]$, define

$$L(t) = \frac{1}{2(b-a)}\int_a^b \left[f(ta+(1-t)x) + f(tb+(1-t)x)\right] dx$$

For the function $L(t)$, Huan-nan Shi [21] found a similar result to Theorem 3.1.

Theorem 3.12. *Let $I \subset \mathbb{R}$ be an interval with a non-empty interior point, and define a binary function as follows*

$$P(a,b) = \begin{cases} L(t),\ a,b \in I,\ a \neq b \\ f(a),\ a = b \end{cases}$$

(i) For $\frac{1}{2} \le t \le 1$, if f is convex on I, then $P(a,b)$ is Schur-convex on I^2.
(ii) For $0 \le t \le \frac{1}{2}$, if f is concave on I, then $P(a,b)$ is Schur-concave on I^2.
Let $f : [a,b] \to \mathbb{R}$ be a convex function, and in the interval $[0,1]$ be the following function.

$$G(t) = \frac{1}{2(b-a)} \int_a^b \left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right) + f\left(\frac{1+t}{2}b + \frac{1-t}{2}x\right) \right] dx.$$

For the function $G(t)$, Huan-nan Shi et al. [22] gave the following results.

Theorem 3.13. *Let $I \subset \mathbb{R}$ be an interval with a non-empty interior point, and f be a continuous function on I. For any $t \in [0,1]$, define a binary function as follows*

$$P(a,b) = \begin{cases} G(t),\ a,b \in I,\ a \neq b \\ f(a),\ a = b \end{cases}$$

If f is convex (or concave, resp.) on I, then $P(a,b)$is Schur-convex (or Schur-concave, resp.) on I^2.

Lan He defined the following two mappings L and F in [23].
$L : [a,b] \times [a,b] \to \mathbb{R}$,

$$L(x,y;f,g) = \left[\int_x^y f(t)dt - (y-x) f\left(\frac{x+y}{2}\right) \right] \left[(y-x) g\left(\frac{x+y}{2}\right) - \int_x^y g(t)dt \right]$$

$F : [a,b] \times [a,b] \to \mathbb{R}$,

$$F(x,y;f,g) = g\left(\frac{x+y}{2}\right) \int_x^y f(t)dt - f\left(\frac{x+y}{2}\right) \int_x^y g(t)dt.$$

In [24], Huan-nan Shi studied Schur convexity on $[a,b] \times [a,b] \subset \mathbb{R}^2$ of $L(x,y;f,g)$ and $F(x,y;f,g)$ with (x,y), the following results are obtained.

Theorem 3.14. *Let f and $-g$ be convex functions on $[a,b]$, then $L(x,y;f,g)$ is Schur-convex on $[a,b] \times [a,b] \subset \mathbb{R}^2$.*

Theorem 3.15. *Let f and $-g$ be non-negative convex functions on $[a,b]$, then $F(x,y;f,g)$ is Schur-convex on $[a,b] \times [a,b] \subset \mathbb{R}^2$.*

Huan-nan Shi studied the Schur-convexity of the following two functions in [24]:

$$H_{p,q}(f,g;a,b) = \begin{cases} \frac{M_p(f;a,b)}{M_q(g;a,b)}, & a \neq b; \\ \frac{f(a)}{g(a)}, & a = b. \end{cases} \tag{3.10}$$

$$L_{p,q}(f;g;a,b) = \begin{cases} \left[M_p(f;a,b) - f(\frac{a+b}{2})\right] \cdot \left[g(\frac{a+b}{2}) - M_q(g;a,b)\right], & a \neq b; \\ 0, & a = b, \end{cases} \tag{3.11}$$

where $M_p(f;a,b)$ see Theorem 3.8.

Theorem 3.16. *Let f and g be Lebesgue integrable real functions defined on the interval $I \subset \mathbb{R}$, and their ranges are J_1 and J_2, p and q are strictly increasing continuous functions on J_1 and J_2 respectively, and $M_p(f;a,b) \ge 0$, $M_q(g;a,b) > 0$ and $g\left(\frac{a+b}{2}\right) \neq 0$.*

(*i*) *If $p \circ f$ is convex on I and $q \circ g$ is concave on I, then $H_{p,q}(f,g;a,b)$ is Schur-convex on I^2, and then for $a < b$*

$$\frac{M_p(f;a,b)}{M_q(g;a,b)} \geq \frac{M_p(f;ta+(1-t)b,tb+(1-t)a)}{M_q(g;ta+(1-t)b,tb+(1-t)a)} \geq \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}, \tag{3.12}$$

where $\frac{1}{2} \leq t \leq 1 or\ 0 \leq t \leq \frac{1}{2}$.

(*ii*) *If $p \circ f$ is concave on I and $q \circ g$ is convex on I, then $H_{p,q}(f,g;a,b)$ is Schur-concave on I^2, and the inequality chain (3.12) is reversed.*

Theorem 3.17. *Let f and g be Lebesgue integrable real functions defined on the interval $I \subseteq \mathbb{R}$, and their ranges are J_1 and J_2, respectively, and let $M_p(f;a,b) \geq 0$, $M_q(g;a,b) > 0$, and $g\left(\frac{a+b}{2}\right) \neq 0$. If p,q are strictly increasing continuous real functions on J_1 and J_2 respectively, and $p \circ f$ is convex on I and $q \circ g$ is concave on I, then $L_{p,q}(f,g;a,b)$ is convex on I^2 Schur-convex, and the following chain of inequalities holds:*

$$\frac{M_p(f;a,b)}{M_q(g;a,b)} \geq \frac{M_p(f;a,b)}{2M_q(g;a,b)} + \frac{f\left(\frac{a+b}{2}\right)}{2g\left(\frac{a+b}{2}\right)} \geq \frac{f\left(\frac{a+b}{2}\right)}{2M_q(g;a,b)} + \frac{M_p(f;a,b)}{2g\left(\frac{a+b}{2}\right)} \geq \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \tag{3.13}$$

4. Schur convexity of the integral mean of bivariate convex functions

In 2022, Huan-nan Shi et al. [25] extended Theorem 2.1 to the case of binary convex functions and established some binary mean inequalities.

Theorem 4.1. *Let I be an interval with non-empty interior points on $\mathbb{R}$ and $f(x,y)$ be a continuous function on I^2. If f is convex (or concave, resp.) on I^2, then*

$$F(a,b) = \begin{cases} \frac{1}{(b-a)^2}\int_a^b\int_a^b f(x,y)dxdy, & (a,b) \in I^2,\ a \neq b \\ f(a,a), & (a,b) \in I^2,\ a = b \end{cases} \tag{4.1}$$

Schur-convex (or Schur-concave, resp.)on I^2.

Using the concavity of $f(x,y) = x^s y^{1-s}$ on $\mathbb{R}^2_{++}$ and combining the majorizing relation $\left(\frac{d+c}{2}, \frac{d+c}{2}\right) \prec (c,d) \prec (c+d,0)$, we obtain the following theorem from Theorem 4.1.

Theorem 4.2. *Let $a > 0$ and $b > 0$. If $a \neq b, 0 < r < 1$, then*

$$\frac{(a+b)^{2r-1}}{r(r+1)} \leq S^r_{r+1}(a,b)S^{r-1}_r(a,b) \leq A(a,b), \tag{4.2}$$

where $A(a,b) = \frac{a+b}{2}$ and $S_r(a,b) = \left(\frac{b^r-a^r}{r(b-a)}\right)^{\frac{1}{r-1}}$ are the arithmetic mean and the Stolarsky mean of the positive numbers a and b of the order of r, respectively.

Using the convexity of $f(x,y) = \frac{1}{(x+y)^2}$ on $\mathbb{R}^2_+$ and combining the majorizing relation $\left(\frac{d+c}{2}, \frac{d+c}{2}\right) \prec (c,d)$, we can obtain the following theorem from Theorem 4.1.

Theorem 4.3. *Let $a > 0, b > 0$, then*

$$\log\left(\frac{A(a,b)}{G(a,b)}\right)^2 \geq \left(\frac{a-b}{a+b}\right)^2, \tag{4.3}$$

where $G(a,b) = \sqrt{ab}$ is the geometric mean of the two positive numbers a and b.

Using the convexity of $f(x,y) = \frac{x^2}{2r^2} + \frac{y^2}{2s^2}$ on $\mathbb{R}^2_{++}$ and combining the majorizing relation $\left(\frac{d+c}{2}, \frac{d+c}{2}\right) \prec (c,d)$, we can obtain the following theorem from Theorem 4.1.

Theorem 4.4. *Let $a>0, b>0$, then*

$$H_e(a^2,b^2) \geq A^2(a,b), \tag{4.4}$$

where $H_e(a,b) = \frac{a+\sqrt{ab}+b}{3}$ is the Heronian mean of the two positive numbers a and b.

In 2025, Chowdappa et al. [26] in India discussed the Schur harmonic convexity of $F(a,b)$.

Theorem 4.5. *Let I be an interval on $\mathbb{R}$ with a non-empty interior point and $f(x,y)$ be a continuous function on I^2. If f is convex (or concave, resp.) on I^2, then*

$$F(a,b) = \begin{cases} \frac{1}{(b-a)^2}\int_a^b\int_a^b f(x,y)dxdy, & (a,b)\in I^2,\ a\neq b \\ f(a,a), & (a,b)\in I^2,\ a=b \end{cases} \tag{4.5}$$

Schur-harmonically convex (or Schur-harmonically concave, resp.) on I^2.

5. Schur-convexity of integral mean value of convex function on the co-ordinates

Definition 5.1. Let the two-dimensional interval $\Delta := [a,b]\times[c,d] \subset \mathbb{R}^2$, $a<b$, $c<d$. $f:\Delta\to\mathbb{R}$ is called a convex function on the co-ordinates if for all $y\in[c,d]$ and $x\in[a,b]$, partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u):=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v):=f(u,v)$ is a convex function.

Lemma 5.2. *Every convex function $f:\Delta\to\mathbb{R}$ is convex function on the co-ordinates, but not vice versa.*

In 2023, Shi Huannan and Zhang Jing [27] studied the monotonicity of the integral mean with respect to the upper and lower limits of the integral for convex function on the co-ordinates, Schur geometric convexity, Schur harmonic convexity and Schur power convexity, and introduced some novel and interesting binary mean inequalities as an application.

Theorem 5.3. *Let $I\subset\mathbb{R}$ be an interval with a non-empty interior point, and $f(a,b)$ is continuous on I^2. If $f(a,b)$ is increasing and convex function on the co-ordinates on I^2, then*

(a) $F(a,b)$on I is decreases with respect to a and increases with respect to b;
(b) $F(a,b)$ is Schur-geometrically convex, Schur-harmonically convex, and Schur-power convex on $I^2\subset\mathbb{R}^2_{++}$.

Theorem 5.4. *Let $c>0, d>0$ and $m\neq 0$, then*

$$H_e(c^2,d^2) \geq (M_m(c,d))^2, \tag{5.1}$$

where $H_e(c,d) = \frac{c+\sqrt{cd}+d}{3}$, $M_m(c,d) = \sqrt[m]{\frac{c^m+d^m}{2}}$ are the Heren mean and power mean of the two positive numbers c and d respectively.

Theorem 5.5. *Let $c>0, d>0$ and $m\geq 2$. Then*

$$S_{m+1}(c,d) \geq M_m(c,d), \tag{5.2}$$

Where $S_{m+1}(c,d) = \left(\frac{d^{m+1}-c^{m+1}}{(m+1)(d-c)}\right)^{\frac{1}{m}}$ and $M_m(c,d) = \sqrt[m]{\frac{c^m+d^m}{2}}$ are the Stolarsky mean of $m+1$ order and the power mean of m order of two positive numbers c and d.

In 2021, Safaei and Barani of Iran [28] studied the Schur convexity of functions obtained from convex function on the co-ordinates on a plane square.

Theorem 5.6. *Let $D := [a_1,b_1]\times[a_1,b_1]\subset\mathbb{R}^2$ be a square, $a_1<b_1$, the function $f:D\to\mathbb{R}$ is continuous and has continuous second-order partial derivatives on D°, take $a,b\in(a_1,b_1)$, $a<b$, let $\Delta := [a,b]\times[a,b]$. If f is convex function on the co-ordinates on Δ, then the function $F:\Delta\to\mathbb{R}$ is Schur-convex on Δ, where*

$$F(x,y) = \begin{cases} \frac{1}{(y-x)^2}\int_x^y\int_x^y f(t,s)dtds, & x\neq y, (x,y)\in[a,b], \\ f(x,x), & x=y, (x,y)\in[a,b]. \end{cases} \tag{5.3}$$

Theorem 5.7. *Let $D := [a_1,b_1] \times [a_1,b_1] \subset \mathbb{R}^2$ be a square, $a_1 < b_1$, the function $f : D \to \mathbb{R}$ is continuous and has continuous second-order partial derivatives on D°, take $a,b \in (a_1,b_1)$, $a < b$, and let $\Delta := [a,b] \times [a,b]$. If f is a convex function on the co-ordinates on Δ, the function $G : \Delta \to \mathbb{R}$ is Schur-convex on Δ, where*

$$G(x,y) = \begin{cases} \frac{1}{(y-x)^2} \int_x^y \int_x^y f(t,s)dtds - f\left(\frac{x+y}{2}, \frac{x+y}{2}\right), & x \neq y, \\ 0, & x = y, \end{cases} \tag{5.4}$$

In 2019, Safaei and Barani [29] studied the Schur harmonically convexity of functions derived from harmonically convex function on the co-ordinates on a plane square.

Definition 5.8. Let $\Delta = [a,b] \times [c,d] \subset (0,\infty) \times (0,\infty)$, $a < b$, $c < d$. A function $f : \Delta \to \mathbb{R}$ is called a harmonically convex function on the co-ordinates on Δ if for any $y \in [c,d]$ and $x \in [a,b]$, the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ is harmonically convex.

Theorem 5.6 Let $I \subset (0,\infty)$ be an open interval, and the function $f : I \to \mathbb{R}_{++}$ is continuously differentiable on I. The function $F : I^2 \to \mathbb{R}_{++}$ is defined as

$$F(x,y) = \begin{cases} \frac{xy}{y-x} \int_x^y \int_x^y \frac{f(t)}{t^2} dt & (x,y) \in I, x \neq y, \\ f(x), & x = y \in I. \end{cases} \tag{5.5}$$

Then F is Schur-harmonically convex on I^2 if and only if f is harmonically convex on I.

Theorem 5.9. *Let $I \subset (0,\infty)$ be an open interval, and the function $f : I \to \mathbb{R}_{++}$ be continuously differentiable on I. If the function $G : I^2 \to \mathbb{R}_{++}$is defined as*

$$G(x,y) = \begin{cases} \frac{xy}{y-x} \int_x^y \int_x^y \frac{f(t)}{t^2} dt - f\left(\frac{2xy}{x+y}\right) & (x,y) \in I, x \neq y, \\ 0, & x = y \in I. \end{cases} \tag{5.6}$$

Then G is Schur-harmonically convex on I^2 if and only if f is harmonically convex over I.

Theorem 5.10. *Let $I \subset (0,\infty)$ be an interval, and the function $f : I \to \mathbb{R}_{++}$ is continuously differentiable on I. If the function $H : I^2 \to \mathbb{R}_{++}$ is defined as*

$$H(x,y) = \begin{cases} \frac{f(x)+f(y)}{2} - \frac{xy}{y-x} \int_x^y \int_x^y \frac{f(t)}{t^2} dt & (x,y) \in I, x \neq y, \\ 0, & x = y \in I. \end{cases} \tag{5.7}$$

Then H is Schur harmonically convex on I^2 if and only if f is harmonically convex on I.

Theorem 5.11. *Let $D = [a_1,b_1] \times [a_1,b_1] \subset \mathbb{R}^2_{++}$ be a square, $a_1 < b_1$, and the function $f : D \to \mathbb{R}^2_{++}$ is continuous on D° and has continuous second-order partial derivatives. Take $a,b \in (a_1,b_1)$, $a < b$, and let $\Delta = [a,b] \times [a,b]$. If f is harmonically convex function on the co-ordinates on Δ, then the function $T : \Delta \to \mathbb{R}^2_{++}$ is Schur-harmonically convex on Δ, where*

$$T(x,y) = \begin{cases} \frac{x^2y^2}{(y-x)^2} \int_x^y \int_x^y \frac{f(t,s)}{t^2s^2} dtds & x \neq y, x,y \in [a,b], \\ f(x,x), & x = y, x,y \in [a,b]. \end{cases} \tag{5.8}$$

Theorem 5.12. *Let $D = [a_1,b_1] \times [a_1,b_1] \subset \mathbb{R}^2_{++}$be a square, $a_1 < b_1$, and the function $f : D \to \mathbb{R}^2_{++}$ is continuous on D° and has continuous second-order partial derivatives. Take $a,b \in (a_1,b_1)$, $a < b$, and let $\Delta = [a,b] \times [a,b]$. If f is harmonically convex function on the co-ordinates on Δ, then the function $L : \Delta \to \mathbb{R}^2_{++}$ is Schur-harmonically convex on Δ, where*

$$L(x,y) = \begin{cases} \frac{x^2y^2}{(y-x)^2} \int_x^y \int_x^y \frac{f(t,s)}{t^2s^2} dtds - f\left(\frac{2xy}{x+y}, \frac{2xy}{x+y}\right) & x \neq y, \\ 0, & x = y. \end{cases} \tag{5.9}$$

In 2020, Nozar Safaei, Ali Barani [30] studied the Schur-convexity of integral arithmetic means of co-ordinated convex functions in $\mathbb{R}^3$, obtained the following results.

Theorem 5.13. *Let $D := [a_1,b_1]\times[a_1,b_1]\times[a_1,b_1]$ be a cube in $\mathbb{R}^3$ with $a_1 < b_1$, and the function $f : D \to \mathbb{R}$ is continuous, and has continuous second order partial derivatives on D°. Choose $a,b \in (a_1,b_1)$, with $a < b$, and let $\Delta := [a,b]\times[a;b]\times[a;b]$. Suppose that f is convex on the co-ordinates on Δ, then the function $F : [a,b]\times[a,b] \to \mathbb{R}$ defined by*

$$F(x,y) =: \begin{cases} \frac{1}{(y-x)^3}\int_x^y\int_x^y\int_x^y f(r,s,t)drdsdt, x \neq y, x,y \in [a,b], \\ f(x,x,x), x = y, x,y \in [a,b]. \end{cases} \tag{5.10}$$

is Schur-convex on $[a,b]\times[a,b]$.

Theorem 5.14. *Let $D := [a_1,b_1]\times[a_1,b_1]\times[a_1,b_1]$ be a cube in $\mathbb{R}^3$ with $a_1 < b_1$, and the function $f : D \to \mathbb{R}$ is continuous, and has continuous second order partial derivatives on D°. Choose $a,b \in (a_1,b_1)$, with $a < b$, and let $\Delta := [a,b]\times[a;b]\times[a;b]$. Suppose that f is convex on the co-ordinates on Δ, then the function $G : [a,b]\times[a,b] \to \mathbb{R}$ defined by*

$$G(x,y) =: \begin{cases} \frac{1}{(y-x)^3}\int_x^y\int_x^y\int_x^y f(r,s,t)drdsdt - f\left(\frac{x+y}{2},\frac{x+y}{2},\frac{x+y}{2}\right), x \neq y, x,y \in [a,b], \\ 0, x = y, x,y \in [a,b]. \end{cases} \tag{5.11}$$

is Schur-convex on $[a,b]\times[a,b]$.

Use of AI Tools

AI tools were not employed in generating, analyzing, or interpreting the results.

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