J. Inequal. Math. Anal., 2(1), 2026, 40–50

Journal of Inequalities and Mathematical Analysis

ISSN:3062-3251
Journal Homepage: www.inequalmath.com

Research Article

A Multidimensional Half-Discrete Hardy-Hilbert's Inequality Involving One Partial Sum

Bicheng Yang *

School of Mathematics, Guangong University of Education, Guangzhou, Guangdong 510303, China.

Abstract

In this paper, by using the weight functions, the idea of introduced parameters and the techniques of real analysis. a multidimensional half-discrete Hardy-Hilbert's inequality with the new kernel as $\frac{1}{(u(m)+||y||_{\beta}^{\alpha})^{\lambda}}$ $(\alpha,\lambda>0)$ involving one partial sum is obtained. The equivalent statements of the best value related to parameters are considered, and some corollaries are deduced.

Keywords: Hardy-Hilbert's inequality, integral inequality, partial sum

2020 MSC: 26D15.

1. Introduction

Assuming that $p>1, \frac{1}{p}+\frac{1}{q}=1, a_m, b_n \geq 0, 0<\sum_{m=1}^{\infty} a_m^p<\infty$ and $0<\sum_{n=1}^{\infty} b_n^q<\infty$, we have the following well known Hardy-Hilbert$'$s inequality with the best value $\frac{\pi}{\sin(\pi/p)}$ (cf [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}. \tag{1.1}$$

If $f(x), g(y)\geq 0$, $0<\int_0^{\infty} f^p(x)dx<\infty$ and $0<\int_0^{\infty} g^q(y)dy<\infty$, then we have the integral analogue of (1.1) with the same best value named in Hardy-Hilbert's integral inequality as follows (cf [1], Theorem 316):

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^{\infty}g^q(y)dy\right)^{\frac{1}{q}}. \tag{1.2}$$

In 2006, by means of Euler-Maclaurin summation formula and the techniques of real analysis, Krnić et al. [2] gave an extension of (1.1) as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}}$$

*Corresponding author. Email: bcyang818@163.com
Email address: bcyang818@163.com (Bicheng Yang *)

doi: 10.63286/jima.032

Received: 7 September 2025 Revised: 11 November 2025 Accepted: 15 November 2025

$$< \quad B(\lambda_1,\lambda_2)\left(\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1}b_n^q\right)^{\frac{1}{q}}. \quad (1.3)$$

where $\lambda_1,\lambda_2 \in (0,2], \lambda_1+\lambda_2 = \lambda \in (0,4]$, the constant $B(\lambda_1,\lambda_2)$ is the best value, and

$$B(u,v) = \int_0^{\infty} \frac{t^{u-1}}{(1+t)^{u+v}}dt, u,v > 0 \quad (1.4)$$

is the Beta function.

In 2016-2017, Hong et al. [3, 4] considered several equivalent conditions of the extensions of (1.1) and (1.2) with a few parameters related to the best values. Some other results were provided in [5–7].

In 2019, by means of (1.3) and Abel's partial summation formula, Adiyasuren et al. [8] obtained an extended application of (1.3) involving two partial sums. In 2020, Mo et al. [9] gave an extension of (1.2) involving two upper limit functions. Inequalities (1.1)-(1.2) with their extensions played an important role in analysis and its applications (cf. [10–20]). In 2023, Hong et al. [21] gave a more accurate multidimensional half-discrete Hilbert-type inequality involving one derivative function of m-order, and [22] gave an extended inequality with the same kernel involving one multiple upper limit function. Some dependent results were published by [23–27].

In this paper, following the way of [21, 22], by using the weight functions, the idea of introduced parameters and the techniques of real analysis. a multidimensional half-discrete Hardy-Hilbert's inequality with the new kernel as $\frac{1}{(u(m)+||y||_\beta^\alpha)^\lambda}$ $(\alpha,\lambda > 0)$ involving one partial sum is obtained. The equivalent statements of the best value related to parameters are considered, and some corollaries are deduced.

2. Some lemmas

In what follows, we assume that

(H1). $p > 1$ $(q > 1), \frac{1}{p}+\frac{1}{q} = 1, \alpha,\beta \in \mathbf{R}_+ := (0,\infty)$, $\lambda > 0$, $\lambda_1,\lambda_2 \in (0,\lambda), m_0, n \in \mathbf{N} := \{1,2,\cdots\}$, $u(x), u'(x) > 0, u''(x) \le 0$, $(u(x))^{\lambda_1}u'(x)$ is decreasing in $x \in (m_0-1,\infty)$, with $u(\infty) = \infty$, $\widehat{\lambda}_1 = \frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\widehat{\lambda}_2 = \frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}, a_k \ge 0$, $A_m^{(0)} := a_m, A_m^{(1)} := \sum_{k=m_0}^m a_k$ $(k,m \in \mathbf{N}_{m_0} := \{m_0, m_0+1,\cdots\})$, satisfying $A_m^{(1)} = o(e^{tu(m)})$ $(t > 0; m \to \infty)$, $g(y) \ge 0$ $(y = (y_1,\cdots,y_n) \in \mathbf{R}_+^n$, $||y||_\beta := (\sum_{i=1}^n y_i^\beta)^{\frac{1}{\beta}}$,

$$0 \quad < \quad \sum_{m=m_0}^{\infty} \frac{(u'(m))^{1-p(1-i)}}{(u(m))^{p(i-1+\widehat{\lambda}_1)+1}}(A_m^{(i)})^p < \infty \ (i \in \{0,1\}), \text{ and}$$

$$0 \quad < \quad \int_{\mathbf{R}_+^n} ||y||_\beta^{q(n-\alpha\widehat{\lambda}_2)-n} g^q(y)dy < \infty.$$

Remark 2.1. (i) In view of the assumption H1, since $i \in \{0,1\}, u'(x) > 0$, $(u(x))^{\lambda_1+i-1}u'(x)$ $(= \frac{1}{(u(x))^{1-i}}[(u(x))^{\lambda_1}u'(x)])$ is also decreasing in $x \in (m_0-1,\infty)$.

(ii) For $\gamma \in (0,1), m_0 = 1, u(x) = x^\gamma, x \in (0,\infty)$, $\lambda_1 \in (0,\frac{1}{\gamma}-1]$, $u(x) > 0, u'(x) = \gamma x^{\gamma-1} > 0, u''(x) = \gamma(\gamma-1)x^{\gamma-2} < 0, u(\infty) = \lim_{x\to\infty} x^\gamma = \infty$, $(u(x))^{\lambda_1}u'(x) = \gamma x^{(\lambda_1+1)\gamma-1}$ is obviously decreasing in $x \in (0,\infty)$.

(iii) For $\gamma \in (0,1), m_0 = 2, u(x) = \ln^\gamma x, x \in (1,\infty)$, $\lambda_1 \in (0,\frac{1}{\gamma}-1]$, $u(x) > 0, u'(x) = \frac{\gamma}{x}\ln^{\gamma-1}x > 0, u''(x) < 0, u(\infty) = \lim_{x\to\infty}\ln^\gamma x = \infty$, $(u(x))^{\lambda_1}u'(x) = \frac{\gamma}{x}\ln^{(\lambda_1+1)\gamma-1}x$ is decreasing in $x \in (1,\infty)$.

If $M > 0, \psi(u)$ $(u > 0)$ is a nonnegative measurable function, then we have the following transfer formula (cf. [10], (9.1.5)):

$$\int\cdots\int_{\{y\in\mathbf{R}_+^n; 0<\sum_{i=1}^n(\frac{y_i}{M})^\beta\le 1\}} \psi(\sum_{i=1}^n(\frac{y_i}{M})^\beta)dy_1\cdots dy_n$$

$$= \frac{M^n\Gamma^n(\frac{1}{\beta})}{\beta^n\Gamma(\frac{n}{\beta})}\int_0^1 \psi(u)u^{\frac{n}{\beta}-1}du. \quad (2.1)$$

(i) For $||y||_\beta = M[\sum_{i=1}^n(\frac{y_i}{M})^\beta]^{\frac{1}{\beta}}, \psi(u) = \varphi(Mu^{\frac{1}{\beta}})$, by (2.1), we have

$$\begin{aligned}
&\int_{\mathbf{R}_+^n} \varphi(||y||_\beta)dy \\
= &\lim_{M\to\infty}\int\cdots\int_{\{y\in\mathbf{R}_+^n;0<\sum_{i=1}^n(\frac{y_i}{M})^\beta\le 1\}} \varphi(M[\sum_{i=1}^n(\frac{y_i}{M})^\beta]^{\frac{1}{\beta}})dy_1\cdots dy_n \\
= &\lim_{M\to\infty}\frac{M^n\Gamma^n(\frac{1}{\beta})}{\beta^n\Gamma(\frac{n}{\beta})}\int_0^1 \varphi(Mu^{\frac{1}{\beta}})u^{\frac{n}{\beta}-1}du \\
\overset{v=Mu^{\frac{1}{\beta}}}{=} &\frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})}\int_0^\infty \varphi(v)v^{n-1}dv. \qquad (2.2)
\end{aligned}$$

(ii) If $\varphi(||y||_\beta) = \varphi(Mu^{\frac{1}{\beta}}) = 0$, for $u = \sum_{i=1}^n(\frac{y_i}{M})^\beta < (\frac{b}{M})^\beta$ $(b > 0)$, i.e. $||y||_\beta = Mu^{\frac{1}{\beta}} < b$, then by (2.2), it follows that

$$\begin{aligned}
\int_{\{y\in\mathbf{R}_+^n;||y||_\beta\ge b\}} \varphi(||y||_\beta)dy &= \lim_{M\to\infty}\frac{M^n\Gamma^n(\frac{1}{\beta})}{\beta^n\Gamma(\frac{n}{\beta})}\int_{(\frac{b}{M})^\beta}^1 \varphi(Mu^{\frac{1}{\beta}})u^{\frac{n}{\beta}-1}du \\
&= \frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})}\int_b^\infty \varphi(v)v^{n-1}dv. \qquad (2.3)
\end{aligned}$$

Remark 2.2. For $b = 1, c \in \mathbf{R}_+, \varphi(v) = v^{-\alpha c-n}$ in (2.3), we have

$$\int_{\{y\in\mathbf{R}_+^n;||y||_\beta\ge 1\}} ||y||_\beta^{-\alpha c-n}dy = \int_1^\infty v^{-\alpha c-n}v^{n-1}dv = \frac{\Gamma^n(\frac{1}{\beta})}{\alpha c\beta^{n-1}\Gamma(\frac{n}{\beta})}. \qquad (2.4)$$

Lemma 2.3. *For $s \in (0,\infty), s_1, s_2 \in (0,s), (u(x))^{s_1-1}u'(x)$ is decreasing in (m_0-1,∞), we define the following weight functions:*

$$\omega_s(s_1,y) := ||y||_\beta^{\alpha(s-s_1)}\sum_{m=m_0}^\infty \frac{(u(m))^{s_1-1}u'(m)}{(u(m)+||y||_\beta^\alpha)^s} \ (y\in\mathbf{R}_+^n), \qquad (2.5)$$

$$\varpi_s(s_2,m) := (u(m))^{s-s_2}\int_{\mathbf{R}_+^n}\frac{||y||_\beta^{\alpha s_2-n}dy}{(u(m)+||y||_\beta^\alpha)^s} \ (m\in\mathbf{N}_{m_0}). \qquad (2.6)$$

The following inequality and expression are value:

$$\omega_s(s_1,y) < B(s_1,s-s_1) \ (y\in\mathbf{R}_+^n), \qquad (2.7)$$

$$\varpi_s(s_2,m) = \frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}B(s_2,s-s_2) \ (m\in\mathbf{N}_{m_0}). \qquad (2.8)$$

Proof. In view of the decreasing property of series, setting $v = \frac{u(x)}{||y||_\beta^\alpha}$, we find

$$\begin{aligned}
\omega_s(s_1,y) &< ||y||_\beta^{\alpha(s-s_1)}\int_{m_0-1}^\infty \frac{(u(x))^{s_1-1}u'(x)}{(u(x)+||y||_\beta^\alpha)^s}dx \\
&\le \int_0^\infty \frac{v^{s_1-1}}{(v+1)^s}dv = B(s_1,s-s_1),
\end{aligned}$$

and then we have (2.7).

In (2.2), for $\varphi(v)=\frac{v^{\alpha s_2-n}}{(u(m)+v^{\alpha})^s}$, we have

$$\begin{aligned}\varpi_s(s_2,m) &= \frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})}(u(m))^{s-s_2}\int_0^{\infty}\frac{v^{\alpha s_2-n}}{(u(m)+v^{\alpha})^s}v^{n-1}dv \\ &= \frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})}(u(m))^{s-s_2}\int_0^{\infty}\frac{v^{\alpha s_2-1}}{(u(m)+v^{\alpha})^s}dv \\ &\overset{t=v^{\alpha}/u(m)}{=} \frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\int_0^{\infty}\frac{t^{s_2-1}}{(1+t)^s}dt \\ &= \frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}B(s_2,s-s_2),\end{aligned}$$

and then we have (2.8).

This proves the lemma. □ □

Lemma 2.4. *With regards to the assumption H1, for $t>0$, we have the following inequality:*

$$\sum_{m=m_0}^{\infty}e^{-tu(m)}a_m \leq t^i\sum_{m=m_0}^{\infty}e^{-tu(m)}(u'(m))^iA_m^{(i)}\ (i\in\{0,1\}). \tag{2.9}$$

Proof. For $i=0$, since $a_m=A_m^{(0)}$, (2.9) keeps the form of an equality; for $i=1$, since $A_m^{(1)}e^{-tu(m)}=o(1)$ $(t>0; m\to\infty)$, by Abel's partial summation formula, we find

$$\begin{aligned}\sum_{m=m_0}^{\infty}e^{-tu(m)}a_m &= \lim_{m\to\infty}A_m^{(1)}e^{-tu(m)}+\sum_{m=m_0}^{\infty}A_m^{(1)}(e^{-tu(m)}-e^{-tu(m+1)}) \\ &= \sum_{m=m_0}^{\infty}A_m^{(1)}(e^{-tu(m)}-e^{-tu(m+1)}).\end{aligned}$$

We set function $f(x):=e^{-tu(x)}, x\in(m_0-1,\infty)$. Then we find

$$f'(x):=-te^{-tu(x)}u'(x)=-th(x),$$

where $h(x)=e^{-tu(x)}u'(x)$ is decreasing in (m_0-1,∞), in view of $u(x),u'(x)>0$ and $u''(x)\leq 0$. By the differentiation mid-value theorem, there exists a $\theta_m\in(0,1)$, such that

$$\begin{aligned}\sum_{m=m_0}^{\infty}e^{-tu(m)}a_m &= -\sum_{m=m_0}^{\infty}A_m^{(1)}[f(m+1)-f(m)] \\ &= -\sum_{m=m_0}^{\infty}A_m^{(1)}f'(m+\theta_m)=t\sum_{m=m_0}^{\infty}A_m^{(1)}h(m+\theta_m) \\ &\leq t\sum_{m=m_0}^{\infty}h(m)A_m^{(1)}=t\sum_{m=m_0}^{\infty}e^{-tu(m)}u'(m)A_m^{(1)},\end{aligned}$$

and then we have (2.9).

This proves the lemma. □ □

Lemma 2.5. *With regards to the assumption H1, for $i\in\{0,1\}$, we have the following inequality:*

$$I_i\ :\ =\int_{\mathbf{R}_+^n}\sum_{m=m_0}^{\infty}\frac{(u'(m))^iA_m^{(i)}g(y)}{(u(m)+||y||_{\beta}^{\alpha})^{\lambda+i}}dy$$

$$
\begin{aligned}
&< \left(\frac{\Gamma^n(\frac{1}{\beta})B(\lambda_2,\lambda+i-\lambda_2)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1+i,\lambda-\lambda_1) \\
&\times \left[\sum_{m=m_0}^{\infty}\frac{(u'(m))^{1-p(1-i)}(A_m^{(i)})^p}{(u(m))^{p(i-1+\widehat{\lambda}_1)+1}}\right]^{\frac{1}{p}} \left[\int_{\mathbf{R}_+^n}||y||_\beta^{q(n-\alpha\widehat{\lambda}_2)-n}g^q(y)dy\right]^{\frac{1}{q}}. \qquad (2.10)
\end{aligned}
$$

Proof. By Hōlder's inequality (cf. [28]), we have

$$
\begin{aligned}
I_i &= \int_{\mathbf{R}_+^n}\sum_{m=m_0}^{\infty}\frac{1}{(u(m)+||y||_\beta^\alpha)^{\lambda+i}}\left[\frac{(u(m))^{(1-\lambda_1-i)/q}A_m^{(i)}}{||y||_\beta^{(n-\alpha\lambda_2)/p}(u'(m))^{-i+(1/q)}}\right] \\
&\quad\times\left[\frac{||y||_\beta^{(n-\alpha\lambda_2)/p}g(y)}{(u(m))^{(1-\lambda_1-i)/q}(u'(m))^{-1/q}}\right]dy \\
&\le \left\{\sum_{m=m_0}^{\infty}\int_{\mathbf{R}_+^n}\frac{1}{(u(m)+||y||_\beta^\alpha)^{\lambda+i}}\frac{(u(m)^{(1-\lambda_1-i)(p-1)}(A_m^{(i)})^p}{||y||_\beta^{n-\alpha\lambda_2}(u'(m))^{-pi+p-1}}dy\right\}^{\frac{1}{p}} \\
&\quad\times\left\{\int_{\mathbf{R}_+^n}\sum_{m=m_0}^{\infty}\frac{1}{(u(m)+||y||_\beta^\alpha)^{\lambda+i}}\frac{||y||_\beta^{(n-\alpha\lambda_2)(q-1)}g^q(y)}{(u(m))^{1-\lambda_1-i}(u'(m))^{-1}}dy\right\}^{\frac{1}{q}} \\
&= \left\{\sum_{m=m_0}^{\infty}[(u(m))^{\lambda+i-\lambda_2}\int_{\mathbf{R}_+^n}\frac{||y||_\beta^{\alpha\lambda_2-n}dy}{(u(m)+||y||_\beta^\alpha)^{\lambda+i}}]\frac{(u'(m))^{1-p+pi}(A_m^{(i)})^p}{(u(m))^{p(i-1+\widehat{\lambda}_1)+1}}\right\}^{\frac{1}{p}} \\
&\quad\times\left\{\int_{\mathbf{R}_+^n}[||y||_\beta^{\alpha(\lambda-\lambda_1)}\sum_{m=m_0}^{\infty}\frac{(u(m)^{\lambda_1+i-1}u'(m)}{(u(m)+||y||_\beta^\alpha)^{\lambda+i}}]||y||_\beta^{q(n-\alpha\widehat{\lambda}_2)-n}g^q(y)dy\right\}^{\frac{1}{q}} \\
&= \left[\sum_{m=m_0}^{\infty}\varpi_{\lambda+i}(\lambda_2,m)\frac{(u'(m))^{1-p+pi}}{(u(m))^{p(i-1+\widehat{\lambda}_1)+1}}(A_m^{(i)})^p\right]^{\frac{1}{p}} \\
&\quad\times\left[\int_{\mathbf{R}_+^n}\omega_{\lambda+i}(\lambda_1+i,y)||y||_\beta^{q(n-\alpha\widehat{\lambda}_2)-n}g^q(y)dy\right]^{\frac{1}{q}}. \qquad (2.11)
\end{aligned}
$$

In (2.11), by (2.7), (2.8) and Remark 1(i), for $s=\lambda+i>0$, $s_1=\lambda_1+i\in(i,\lambda+i)(\subset(0,\lambda+i))$, $s_2=\lambda_2\in(0,\lambda+i)$, since $(u(x))^{s_1-1}u'(x)=(u(x))^{(\lambda_1+i)-1}u'(x)$ is decreasing in (m_0-1,∞), in view of H1, we have (2.10).

This proves the lemma. □ □

3. Main results

Theorem 3.1. *With regards to the assumption H1, for $i\in\{0,1\}$, we have the following half-discrete multidimensional Hardy-Hilbert's inequality involving one partial sum:*

$$
I \ : \ = \int_{\mathbf{R}_+^n}\sum_{m=m_0}^{\infty}\frac{a_m g(y)}{(u(m)+||y||_\beta^\alpha)^\lambda}dy
$$

$$< \lambda^i \left(\frac{\Gamma^n(\frac{1}{\beta})B(\lambda_2, \lambda + i - \lambda_2)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1 + i, \lambda - \lambda_1)$$

$$\times \left[\sum_{m=m_0}^{\infty} \frac{(u'(m))^{1-p+pi}(A_m^{(i)})^p}{(u(m))^{p(i-1+\widehat{\lambda}_1)+1}} \right]^{\frac{1}{p}} \left[\int_{\mathbf{R}_+^n} ||y||_\beta^{q(n-\alpha\widehat{\lambda}_2)-n} g^q(y) dy \right]^{\frac{1}{q}}. \quad (3.1)$$

In particular, for $\lambda = \lambda_1 + \lambda_2$, we have

$$0 < \sum_{m=m_0}^{\infty} \frac{(u'(m))^{1-p+pi}}{(u(m))^{p(i-1+\lambda_1)+1}} (A_m^{(i)})^p < \infty,$$

$$0 < \int_{\mathbf{R}_+^n} ||y||_\beta^{q(n-\alpha\lambda_2)-n} g^q(y) dy < \infty,$$

and the following inequality:

$$\int_{\mathbf{R}_+^n} \sum_{m=m_0}^{\infty} \frac{a_m g(y) dy}{(u(m) + ||y||_\beta^\alpha)^\lambda} < \lambda^i \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B(\lambda_1 + i, \lambda_2)$$

$$\times \left[\sum_{m=m_0}^{\infty} \frac{(u'(m))^{1-p+pi}(A_m^{(i)})^p}{(u(m))^{p(i-1+\lambda_1)+1}} \right]^{\frac{1}{p}} \left[\int_{\mathbf{R}_+^n} ||y||_\beta^{q(n-\alpha\lambda_2)-n} g^q(y) dy \right]^{\frac{1}{q}}. \quad (3.2)$$

Proof. By the following expression of the Gamma function:

$$\frac{1}{(u(m) + ||y||_\beta^\alpha)^\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-(u(m)+||y||_\beta^\alpha)t} dt,$$

(2.9) and Lebesgue term by term theorem (cf. [29]), we have

$$\begin{aligned}
I &= \frac{1}{\Gamma(\lambda)} \int_{\mathbf{R}_+^n} \sum_{m=m_0}^{\infty} a_m g(y) \left[\int_0^\infty t^{\lambda-1} e^{-(u(m)+||y||_\beta^\alpha)t} dt \right] dy \\
&= \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} \left(\sum_{m=m_0}^{\infty} e^{-xu(m)} a_m \right) \left(\int_{\mathbf{R}_+^n} e^{-||y||_\beta^\alpha t} g(y) dy \right) dt \\
&\leq \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} (t^i \sum_{m=m_0}^{\infty} e^{-xu(m)} (u'(m))^i A_m^{(i)}) (\int_{\mathbf{R}_+^n} e^{-||y||_\beta^\alpha t} g(y) dy) dt \\
&= \frac{1}{\Gamma(\lambda)} \int_{\mathbf{R}_+^n} \sum_{m=m_0}^{\infty} (u'(m))^i A_m^{(i)} g(y) \left[\int_0^\infty t^{(\lambda+i)-1} e^{-(u(m)+||y||_\beta^\alpha)t} dt \right] dy \\
&= \frac{\Gamma(\lambda+i)}{\Gamma(\lambda)} \int_{\mathbf{R}_+^n} \sum_{m=m_0}^{\infty} \frac{u'(m))^i A_m^{(i)} g(y)}{(u(m) + ||y||_\beta^\alpha)^{\lambda+i}} dy = \lambda^i I_i.
\end{aligned}$$

Then by (2.10), we have (3.1). For $\lambda = \lambda_1 + \lambda_2$ in (3.1), we have (3.2).

This proves the theorem. □ □

Theorem 3.2. *With regards to the assumption H1, if $i \in \{0, 1\}$, $\lambda_1 + \lambda_2 = \lambda$, then the constant factor*

$$\lambda^i \left(\frac{\Gamma^n(\frac{1}{\beta})B(\lambda_2, \lambda + i - \lambda_2)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1 + i, \lambda - \lambda_1)$$

in (3.1) is the best value.

Proof. We need to prove that the constant factor in (3.2) is the best value for $i \in \{0,1\}$. For any $0 < \varepsilon < \min\{p\lambda_1, q\lambda_2\}$, we set

$$\begin{aligned}
\widetilde{A}_m^{(0)} &= \widetilde{a}_m := (u(m))^{\lambda_1 - \frac{\varepsilon}{p} - 1} u'(m), \\
\widetilde{A}_m^{(1)} &= \sum_{k=m_0}^{m} \widetilde{a}_k = \sum_{k=m_0}^{m} (u(k))^{(\lambda_1 - \frac{\varepsilon}{p}) - 1} u'(k), m \in \mathbf{N}_{m_0}, \\
\widetilde{g}(y) &:= \begin{cases} 0, ||y||_\beta < 1 \\ ||y||_\beta^{\alpha(\lambda_2 - \frac{\varepsilon}{q}) - n}, ||y||_\beta \geq 1 \end{cases}.
\end{aligned}$$

Since both $(u(x))^{(\lambda_1 - \frac{\varepsilon}{p}) - 1} u'(x) (= (u(x))^{-\frac{\varepsilon}{p} - 1} [(u(x))^{\lambda_1} u'(x)])$ and $(u(x))^{-\varepsilon - 1} u'(x)$ $(= (u(x))^{-\lambda_1 - \varepsilon - 1} [(u(x))^{\lambda_1} u'(x)])$ are strictly decreasing in $(m_0 - 1, \infty)$, we find

$$\widetilde{A}_m^{(1)} < \int_{m_0 - 1}^{m} (u(x))^{(\lambda_1 - \frac{\varepsilon}{p}) - 1} u'(x) dx \leq \frac{1}{\lambda_1 - \frac{\varepsilon}{p}} (u(m))^{\lambda_1 - \frac{\varepsilon}{p}},$$

and then for $i \in \{0,1\}$, it follows that

$$\begin{aligned}
\widetilde{A}_m^{(i)} &\leq \frac{1}{(\lambda_1 - \frac{\varepsilon}{p})^i} (u(m))^{\lambda_1 - \frac{\varepsilon}{p} + i - 1} (u'(m))^{1-i} \ (m \in \mathbf{N}_{m_0}), \text{ and} \\
& \sum_{m=m_0}^{\infty} \frac{(u'(m))^{1-p+pi}}{(u(m))^{p(i-1+\lambda_1)+1}} (\widetilde{A}_m^{(i)})^p \\
&\leq \frac{1}{(\lambda_1 - \frac{\varepsilon}{p})^{ip}} \left[\frac{u'(m_0)}{(u(m_0))^{\varepsilon+1}} + \sum_{m=m_0+1}^{\infty} \frac{u'(m)}{(u(m))^{\varepsilon+1}} \right] \\
&< \frac{1}{(\lambda_1 - \frac{\varepsilon}{p})^{ip}} \left[\frac{u'(m_0)}{(u(m_0))^{\varepsilon+1}} + \int_{m_0}^{\infty} \frac{u'(x)}{(u(x))^{\varepsilon+1}} dx \right] \\
&= \frac{1}{(\lambda_1 - \frac{\varepsilon}{p})^{ip}} \left[d + \frac{1}{\varepsilon (u(m_0))^{\varepsilon}} \right] \ (d := \frac{u'(m_0)}{(u(m_0))^{\varepsilon+1}}).
\end{aligned}$$

If there exists a positive constant M, with

$$M \leq \lambda^i \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B(\lambda_1 + i, \lambda_2),$$

such that (3.2) is valid when we replace the constant factor by M, then in particular, by (2.4) (for $c = \varepsilon$), we have

$$\begin{aligned}
\widetilde{I} &:= \int_{\mathbf{R}_+^n} \sum_{m=m_0}^{\infty} \frac{\widetilde{a}_m \widetilde{g}(y)}{(u(m) + ||y||_\beta^\alpha)^\lambda} dy \\
&< M \left[\sum_{m=m_0}^{\infty} \frac{(u'(m))^{1-p+pi} (\widetilde{A}_m^{(i)})^p}{(u(m))^{p(i-1+\lambda_1)+1}} \right]^{\frac{1}{p}} \left[\int_{\mathbf{R}_+^n} ||y||_\beta^{q(n - \alpha\lambda_2) - n} \widetilde{g}^q(y) dy \right]^{\frac{1}{q}} \\
&\leq M \left(\sum_{m=m_0}^{\infty} \frac{u'(m)}{(u(m))^{\varepsilon+1}} \right)^{\frac{1}{p}} \left(\int_{\{y \in \mathbf{R}_+^n; ||y||_\beta^\alpha \geq 1\}} ||y||_\beta^{-\alpha\varepsilon - n} dy \right)^{\frac{1}{q}} \\
&\leq \frac{M}{\varepsilon (\lambda_1 - \frac{\varepsilon}{p})^i} \left[\varepsilon d + \frac{1}{(u(m_0))^{\varepsilon}} \right]^{\frac{1}{p}} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{q}}.
\end{aligned}$$

By (2.3), we have

$$\begin{aligned}
\widetilde{I} &= \sum_{m=m_0}^{\infty}(u(m))^{\lambda_1-\frac{\varepsilon}{p}-1}u'(m)\left[\int_{\{y\in\mathbf{R}_+^n;||y||_\beta\geq 1\}}\frac{||y||_\beta^{\alpha(\lambda_2-\frac{\varepsilon}{q})-n}}{(u(m)+||y||_\beta^\alpha)^\lambda}dy\right]\\
&= \frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})}\sum_{m=m_0}^{\infty}(u(m))^{\lambda_1-\frac{\varepsilon}{p}-1}u'(m)\int_1^\infty\frac{v^{\alpha(\lambda_2-\frac{\varepsilon}{q})-1}}{(u(m)+v^\alpha)^\lambda}dv\\
&= \frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\sum_{m=m_0}^{\infty}(u(m))^{-\varepsilon-1}u'(m)\int_{\frac{1}{u(m)}}^\infty\frac{t^{\lambda_2-\frac{\varepsilon}{q}-1}}{(1+t)^\lambda}dt\\
&> \frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\left[B(\lambda_1+\frac{\varepsilon}{q},\lambda_2-\frac{\varepsilon}{q})\sum_{m=m_0}^{\infty}(u(m))^{-\varepsilon-1}u'(m)\right.\\
&\quad\left.-\sum_{m=m_0}^{\infty}(u(m))^{-\varepsilon-1}u'(m)\int_0^{\frac{1}{u(m)}}t^{\lambda_2-\frac{\varepsilon}{q}-1}dt\right]\\
&\geq \frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\left[B(\lambda_1+\frac{\varepsilon}{q},\lambda_2-\frac{\varepsilon}{q})\int_{m_0}^\infty(u(x))^{-\varepsilon-1}u'(x)dx\right.\\
&\quad\left.-\frac{1}{\lambda_2-\frac{\varepsilon}{q}}\sum_{m=m_0}^{\infty}(u(m))^{-\lambda_2-\frac{\varepsilon}{p}-1}u'(m)\right]\\
&= \frac{\Gamma^n(\frac{1}{\beta})}{\varepsilon\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\left[B(\lambda_1+\frac{\varepsilon}{q},\lambda_2-\frac{\varepsilon}{q})(u(m_0))^{-\varepsilon}-\varepsilon O(1)\right].
\end{aligned}$$

Based on the above results, we have the following inequality

$$\begin{aligned}
&\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\left[B(\lambda_1+\frac{\varepsilon}{q},\lambda_2-\frac{\varepsilon}{q})(u(m_0))^{-\varepsilon}-\varepsilon O(1)\right]\\
<\ &\varepsilon\widetilde{I}<\frac{M}{(\lambda_1-\frac{\varepsilon}{p})^i}\left[\varepsilon d+\frac{1}{(u(m_0))^\varepsilon}\right]^{\frac{1}{p}}\left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{q}}.
\end{aligned}$$

For $\varepsilon\to 0^+$, in view of the continuity of the Beta function, we have

$$\frac{B(\lambda_1,\lambda_2)\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\leq\frac{M}{\lambda_1^i}\left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{q}}.$$

Since $i\in(0,1\}$, $\lambda^iB(\lambda_1+i,\lambda_2)=\lambda_1^iB(\lambda_1,\lambda_2)$, it follows that

$$\begin{aligned}
&\lambda^i\left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}}B(\lambda_1+i,\lambda_2)\\
=\ &\lambda_1^i\left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}}B(\lambda_1,\lambda_2)\leq M.
\end{aligned}$$

Therefore,

$$M = \lambda^i \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B(\lambda_1 + i, \lambda_2)$$

is the best value of (3.2) (namely, for $\lambda_1 + \lambda_2 = \lambda$ in (3.1)).

This proves the theorem. □ □

Theorem 3.3. *With regards to the assumption H1, if the constant factor in (3.1) is the best possible, then for $\lambda - \lambda_1 - \lambda_2 \leq 0$, we have $\lambda_1 + \lambda_2 = \lambda$.*

Proof. For $\widehat{\lambda}_1 = \frac{\lambda-\lambda_1-\lambda_2}{p} + \lambda_1, \widehat{\lambda}_2 = \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, we find $\widehat{\lambda}_1 + \widehat{\lambda}_2 = \lambda$ with $0 < \widehat{\lambda}_1, \widehat{\lambda}_2 < \lambda$. For $\lambda - \lambda_1 - \lambda_2 \leq 0$, we observe that

$$(u(x))^{\widehat{\lambda}_1} u'(x) = (u(x))^{\frac{\lambda-\lambda_1-\lambda_2}{p}} [(u(x))^{\lambda_1} u'(x)]$$

is decreasing in $(m_0 - 1, \infty)$.

By Hölder's inequality (cf. [28]), we obtain

$$\begin{aligned}
& B(\widehat{\lambda}_1 + i, \widehat{\lambda}_2) \\
= & \int_0^\infty \frac{u^{\widehat{\lambda}_1+i-1} du}{(1+u)^{\lambda+i}} = \int_0^\infty \frac{(u^{\frac{\lambda-\lambda_2+i-1}{p}})(u^{\frac{\lambda_1+i-1}{q}})}{(1+u)^{\lambda+i}} du \\
\leq & \left[\int_0^\infty \frac{u^{\lambda-\lambda_2+i-1}}{(1+u)^{\lambda+i}} du \right]^{\frac{1}{p}} \left[\int_0^\infty \frac{u^{\lambda_1+i-1}}{(1+u)^{\lambda+i}} du \right]^{\frac{1}{q}} \\
= & \left[\int_0^\infty \frac{v^{\lambda_2-1}}{(1+v)^{\lambda+i}} dv \right]^{\frac{1}{p}} \left[\int_0^\infty \frac{u^{\lambda_1+i-1}}{(1+u)^{\lambda+i}} du \right]^{\frac{1}{q}} \\
= & B^{\frac{1}{p}}(\lambda_2, \lambda + i - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + i, \lambda - \lambda_1).
\end{aligned} \tag{3.3}$$

Since the constant factor

$$\lambda^i \left(\frac{\Gamma^n(\frac{1}{\beta}) B(\lambda_2, \lambda + i - \lambda_2)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1 + i, \lambda - \lambda_1)$$

in (3.1) is the best value. comparing with the constant factors in (3.1) and (3.2) (for $\lambda_1 = \widehat{\lambda}_1, \lambda_2 = \widehat{\lambda}_2$), we have

$$\begin{aligned}
& \lambda^i \left(\frac{\Gamma^n(\frac{1}{\beta}) B(\lambda_2, \lambda + i - \lambda_2)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1 + i, \lambda - \lambda_1) \\
\leq & \lambda^i \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B(\widehat{\lambda}_1 + i, \widehat{\lambda}_2),
\end{aligned}$$

namely,

$$B(\widehat{\lambda}_1 + i, \widehat{\lambda}_2) \geq B^{\frac{1}{p}}(\lambda_2, \lambda + i - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + i, \lambda - \lambda_1).$$

Hence, (3.3) keeps the form of equality. The necessary and sufficient condition for taking an equal sign is that there exist constants A and B, such that they are not both zero, and (cf. [28]) $Au^{\lambda-\lambda_2+i-1} = Bu^{\lambda_1+i-1}$ a.e. in $\mathbf{R}_+$. Assuming that $A \neq 0$, we have $u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A}$ a.e. in $\mathbf{R}_+$. It follows that $\lambda - \lambda_2 - \lambda_1 = 0$, and then $\lambda_1 + \lambda_2 = \lambda$.

This proves the theorem. □

Remark 3.4. For $\gamma \in (0, 1)$, in view of Remark 1, both $u_1(x) = x^\gamma$, $(x \in (0, \infty))$ and $u_2(x) = \ln^\gamma x$ $(x \in (1, \infty))$ satisfy for using Theorem 1-3.

□

Corollary 3.5. *For $i = 0$ in (3.1), we have the following inequality:*

$$I \quad : \quad = \int_{\mathbf{R}_+^n} \sum_{m=m_0}^{\infty} \frac{a_m g(y)}{(u(m) + ||y||_\beta^\alpha)^\lambda} dy$$

$$< \left(\frac{\Gamma^n(\frac{1}{\beta}) B(\lambda_2, \lambda - \lambda_2)}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$$

$$\times \left[\sum_{m=m_0}^{\infty} \frac{(u'(m))^{1-p} a_m^p}{(u(m))^{p(\widehat{\lambda}_1 - 1)+1}} \right]^{\frac{1}{p}} \left[\int_{\mathbf{R}_+^n} ||y||_\beta^{q(n-\alpha\widehat{\lambda}_2)-n} g^q(y) dy \right]^{\frac{1}{q}}. \quad (3.4)$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$, we have

$$\int_{\mathbf{R}_+^n} \sum_{m=m_0}^{\infty} \frac{a_m g(y)}{(u(m) + ||y||_\beta^\alpha)^\lambda} dy < \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B(\lambda_1, \lambda_2)$$

$$\times \left[\sum_{m=m_0}^{\infty} \frac{(u'(m))^{1-p} a_m^p}{(u(m))^{p(\lambda_1 - 1)+1}} \right]^{\frac{1}{p}} \left[\int_{\mathbf{R}_+^n} ||y||_\beta^{q(n-\alpha\lambda_2)-n} g^q(y) dy \right]^{\frac{1}{q}}. \quad (3.5)$$

Corollary 3.6. *If $\lambda_1 + \lambda_2 = \lambda$, then the constant factor*

$$\left(\frac{\Gamma^n(\frac{1}{\beta}) B(\lambda_2, \lambda - \lambda_2)}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$$

in (3.4) is the best value. On the other hand, if the same constant factor in (3.4) is the best value, then for $\lambda - \lambda_1 - \lambda_2 \leq 0$, we have $\lambda_1 + \lambda_2 = \lambda$.

Remark 3.7. Inequality (3.1) (resp. (3.2)) is an extended application of (3.4) (resp. (3.5)).

4. Conclusion

In this paper, following the way of [21, 22], by means of the weight functions, the idea of introduced parameters, the techniques of real analysis and Abel's partial summation formula, a multidimensional half-discrete Hardy-Hilbert's inequality with the new kernel as $\frac{1}{(u(m)+||y||_\beta^\alpha)^\lambda}$ $(\alpha, \lambda > 0)$ involving one partial sum is obtained in Theorem 1. The equivalent statements of the best value related to several parameters in the new inequality are given in Theorem 2 - 3. Some particular results are provided in Corollary 1- 2.

Acknowledgements

This work is supported by the National Science Foundation of China (No. 61772140).

Use of AI Tools

AI tools were not employed in generating, analyzing, or interpreting the results.

References

[1] G. H. Hardy, J. E. Littlewood, and G. Polya, Inequalities, Cambridge University Press, Cambridge, 1934.
[2] M. Krnić and J. Pečarić, *Extension of Hilbert's inequality, Journal of Mathematical Analysis and Applications*, 324(1), 2006, 150–160. DOI: 10.1016/j.jmaa.2005.11.069
[3] Y. Hong and Y. M. Wen, *A necessary and sufficient condition of that Hilbert type series inequality with homogeneous kernel has the best constant factor*, Chinese Annals of Mathematics, 37A(3), 2016, 329–336.
[4] Y. Hong, *On the structure character of Hilbert's type integral inequality with homogeneous kernel and applications*, Journal of Jilin University Science Edition, 55(2), 2017, 189-194.
[5] Y. Hong, Q. L. Huang, B. C. Yang, and J. Q. Liao, *The necessary and sufficient conditions for the existence of a kind of Hilbert-type multiple integral inequality with the non-homogeneous kernel and its applications*, Journal of Inequalities and Applications, 2017, Art. 316, pp. 1–12. DOI: 10.1186/s13660-017-1592-8
[6] Q. Chen, B. He, Y. Hong, and Z. Li, *Equivalent parameter conditions for the validity of half-discrete Hilbert-type multiple integral inequality with generalized homogeneous kernel*, Journal of Function Spaces, 2020(1), Art. 7414861, pp. 1–6. DOI: 10.1155/2020/7414861
[7] B. He, Y. Hong, and Z. Li, *Conditions for the validity of a class of optimal Hilbert type multiple integral inequalities with non-homogeneous kernel*, Journal of Inequalities and Applications, 2021, Art. 64, pp. 1–12. DOI: 10.1186/s13660-021-02593-z
[8] V. Adiyasuren, T. Batbold, and E. A. Azar, *A new discrete Hilbert-type inequality involving partial sums*, Journal of Inequalities and Applications, 2019 Art. 127, pp. 1–6. DOI: 10.1186/s13660-019-2087-6
[9] H. M. Mo and B. C. Yang, *On a new Hilbert-type integral involving the upper limit functions*, Journal of Inequalities and Applications, 2020, Art. 5, pp. 1–12. DOI: 10.1186/s13660-019-2280-7
[10] B. C. Yang, *The Norm of Operator and Hilbert-type Inequality*, Science Press, Beijing, 2009.
[11] M. Krnić and J. Pečarić, *General Hilbert's and Hardy's inequalities*, Mathematical Inequalities & Applications, 8(1), 2005, 29–51. DOI: 10.7153/mia-08-04
[12] I. Perić and P. Vuković, *Multiple Hilbert's type inequalities with a homogeneous kernel*, Banach Journal of Mathematical Analysis, 5(2), 2011, 33–43. DOI: 10.15352/bjma/1313363000
[13] Q. Huang, *A new extension of Hardy-Hilbert-type inequality*, Journal of Inequalities and Applications, 2015, Art. 397, pp. 1–12. DOI: 10.1186/s13660-015-0918-7
[14] B. He, *A multiple Hilbert-type discrete inequality with a new kernel and best possible constant factor*, Journal of Mathematical Analysis and Applications, 2015, 431(2), 902–990. DOI: 10.1016/j.jmaa.2015.06.019
[15] J. S. Xu, *Hardy-Hilbert's inequalities with two parameters*, Advances in Mathematics, 36(2), 2007, 63–76.
[16] Z. T. Xie, Z. Zeng, and Y. F. Sun, *A new Hilbert-type inequality with the homogeneous kernel of degree -2*, Advances and Applications in Mathematical Sciences, 12(7), 2013, 391–401.
[17] Z. Zeng, K. Raja Rama Gandhi, and Z. T. Xie, *A new Hilbert-type inequality with the homogeneous kernel of degree -2 and with the integral*, Bulletin of Mathematical Sciences and Applications, 3(1), 2014, 11–20.
[18] D. M. Xin, *A Hilbert-type integral inequality with the homogeneous kernel of zero degree*, Mathematical Theory and Applications, 30(2), 2010, 70–74.
[19] L. E. Azar, *The connection between Hilbert and Hardy inequalities*, Journal of Inequalities and Applications, 2013, Art. 452, pp. 1–12. DOI: 10.1186/1029-242X-2013-452
[20] V. Adiyasuren, T. Batbold, and M. Krnić, *Hilbert–type inequalities involving differential operators, the best constants and applications*, Mathematical Inequalities & Applications, 18(1), 2015, 111–124. DOI: 10.7153/mia-18-07
[21] Y. Hong, Y. R. Zhong, and B. C. Yang, *On a more accurate half-discrete multidimensional Hilbert-type inequality involving one derivative function of m-order*, Journal of Inequalities and Applications, 2023, Art. 74, pp. 1–15. DOI: 10.1186/s13660-023-02980-8
[22] Y. Hong, Y. R. Zhong, and B. C. Yang, *A more accurate half-discrete multidimensional Hilbert-type inequality involving one multiple upper limit function*, Axioms, 2023, 12(2), Art. 211, pp. 1–13. DOI: 10.3390/axioms12020211
[23] B. C. Yang and J. Q. Liao, *Parameterized multidimensional Hilbert-type inwqualities*, Scientific Rescarch Publisheng, 2020, USA.
[24] Y. Li, Y. Zhong, and B. C. Yang, *Equivalent statements of two multidimensional Hilbert-type integral inequalities involving one with parameters*, Axioms, 2023, 12(10), Art. 956, pp. 1–13. DOI: 10.3390/axioms12100956
[25] J. Q. Liao and B. C. Yang, *A new reverse half-discrete Hilbert-type inequality with one partial sum involving one derivative function of higher order*, Open Mathematics, 2023, 21, Art. 20230139, pp. 1–12. DOI:10.1515/math-2023-0139
[26] A. Z. Wang and B. C. Yang, *A new half-discrete Hilbert-type inequality involving one multiple upper limit function and one partial sums* Annals Mathematics, 45A(1), 2024, 25–38. DOI:10.16205/j.cnki.cama.2024.0003
[27] Z. H. Zeng and B. C. Yang, *A Hardy-Hilbert-type integral inequality involving the derivative functions of n-order*, Journal of South China Normal University (Natural Science Editon), 56(4), 2024, 123–128. DOI:10.1515/math-2023-0139
[28] J. C. Kuang, *Applied inequalities*, Shandong Science and Technology Press, Jinan, 2004.
[29] J. C. Kuang, *Real and functional analysis* (Continuation), Vol. 2, Higher Education Press, Beijing, 2015.