

Research Article

## Schur Convexity of Functional Bonferroni Means

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### Abstract

In this paper, we introduce a functional extension of the classical Bonferroni mean. Using tools from majorization theory together with differential criteria for Schur convexity, we establish sufficient conditions under which the functional Bonferroni mean and the functional generalized Bonferroni harmonic mean are Schur convex, Schur concave, or Schur harmonically convex. As applications, we derive separation inequalities between these two means, obtain several new integral inequalities, and prove norm inequalities for functional Bonferroni means.

**Keywords:** Bonferroni means, functional Bonferroni means, Schur convexity, majorization, norm inequalities

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### 1. Introduction

Throughout this paper, let  $\mathbb{R}^n$  denote the set of real  $n$ -dimensional row vectors. Define

$$\mathbb{R}_+^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\},$$

$$\mathbb{R}_{++}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, \dots, n\}.$$

In particular,  $\mathbb{R}^1, \mathbb{R}_+^1, \mathbb{R}_{++}^1$  are denoted by  $\mathbb{R}, \mathbb{R}_+, \mathbb{R}_{++}$ , respectively.

For  $x = (x_1, \dots, x_n) \in \mathbb{R}_{++}^n$ , the arithmetic mean and harmonic mean are defined by

$$A_n(x) = \frac{1}{n} \sum_{i=1}^n x_i, \quad (1.1)$$

$$H_n(x) = \frac{n}{\sum_{i=1}^n \frac{1}{x_i}}. \quad (1.2)$$

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For  $x = (x_1, \dots, x_n) \in \mathbb{R}_{++}^n$  and  $p, q \in \mathbb{R}_+$ . The Bonferroni mean (BM)[1–3] is defined by

$$B_n^{p,q}(x) = \left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^n x_i^p x_j^q \right)^{\frac{1}{p+q}}. \quad (1.3)$$

This operator plays an important role in information theory and aggregation theory. Several classical means arise as special cases, including the arithmetic mean, geometric mean, and the Muirhead mean.

In particular, if  $q = 0$ , then by (1.3), the BM reduces to the generalized mean operator as follows:

$$B_n^{p,0}(x) = \left[ \frac{1}{n} \sum_{i=1}^n x_i^p \left( \frac{1}{n-1} \sum_{\substack{j=1 \\ j \neq i}}^n x_j^0 \right) \right]^{\frac{1}{p+0}} = \left( \frac{1}{n} \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}}. \quad (1.4)$$

If  $p = 1$ , then by (1.4), BM reduces to the well-known arithmetic mean (AM):

$$B_n^{1,0}(x) = \frac{1}{n} \sum_{i=1}^n x_i. \quad (1.5)$$

When  $p \rightarrow 0$ , by (1.3), BM reduces to the geometric mean (GM), i.e.,

$$\lim_{p \rightarrow 0} B_n^{p,0}(x) = \prod_{i=1}^n x_i^{\frac{1}{n}}. \quad (1.6)$$

For  $n = 2$ , by (1.3), the BM reduces to the Muirhead mean:

$$B_2^{p,q}(x) = \left( \frac{x_1^p x_2^q + x_1^q x_2^p}{2} \right)^{\frac{1}{p+q}}. \quad (1.7)$$

For the study of the Schur convexity of the Muirhead mean, please refer to [2–4].

For  $x = (x_1, \dots, x_n) \in \mathbb{R}_{++}^n$ ,  $p, q \geq 0$  and  $p + q \neq 0$ , the generalized Bonferroni harmonic mean operator is defined (see [5]):

$$BH_n(p, q; x) = \frac{1}{\left( \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^n \frac{1}{x_i^p x_j^q} \right)^{\frac{1}{p+q}}}. \quad (1.8)$$

In 2017, Shi and Wang [6] first studied the Schur convexity, Schur geometric convexity and Schur harmonic convexity of Bonferroni mean. In 2017 and 2019, Shi and Wu [7, 8] respectively discussed the Schur convexity and Schur- $m$  power convexity of geometric Bonferroni mean. In 2022 and 2023, Wang and Wu [9, 10] respectively discussed the Schur- $m$  power convexity of Bonferroni mean.

Motivated by these observations, we introduce functional extensions of the Bonferroni mean and study their Schur-type convexity properties.

Let  $E \subseteq \mathbb{R}$  be nonempty and let  $L^+$  denote the class of positive functions on  $E$ . For  $x = (x_1, \dots, x_n) \in E^n$ ,  $f, g \in L^+$ , and  $p, q \geq 0$  with  $p + q \neq 0$ , define the **functional Bonferroni mean** by

$$B_n(p, q; f, g, x) = \left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^n f^p(x_i) \cdot g^q(x_j) \right)^{\frac{1}{p+q}}. \quad (1.9)$$

This operator satisfies the following properties:

- (i) **Symmetry:**  $B_n(p, q; f, g, x)$  is symmetric in  $x_1, \dots, x_n$ .  
(ii) **Positive homogeneity:** For  $\lambda > 0$ ,

$$B_n(p, q; \lambda f, \lambda g, x) = \lambda B_n(p, q; f, g, x).$$

- (iii) **Normalization:** If  $f(x) \equiv g(x) \equiv 1$ , then  $B_n(p, q; f, g, x) = 1$ .  
(iv) **Monotonicity:** If  $f_1 \geq f_2$  and  $g_1 \geq g_2$  pointwise on  $E$ , then

$$B_n(p, q; f_1, g_1, x) \geq B_n(p, q; f_2, g_2, x).$$

- (v) **Bounds:** If  $a \leq f(x), g(x) \leq b$  for all  $x \in E$ , then

$$a \leq B_n(p, q; f, g, x) \leq b.$$

Accordingly,  $B_n(p, q; f, g, x)$  is a mean of the values  $f(x_1), \dots, f(x_n)$  and is therefore called the *functional Bonferroni mean*.

We also define the **functional generalized Bonferroni harmonic mean** by

$$BH_n(p, q; f, g, x) = \frac{1}{\left(\frac{1}{n^2} \sum_{i,j=1}^n \frac{1}{f^p(x_i)g^q(x_j)}\right)^{\frac{1}{p+q}}}. \quad (1.10)$$

Obviously, if  $f(x) = g(x) = x > 0$ , then by (1.9),  $B_n(p, q; f, g, x)$  reduces to Bonferroni mean, and by (1.10),  $BH_n(p, q; f, g, x)$  reduces to the generalized Bonferroni harmonic mean.

In this paper, we study the Schur convexity of functional Bonferroni mean and functional generalized Bonferroni harmonic mean with  $x$  and give some applications.

We now state our main theorems; all proofs are given in the subsequent section.

**Theorem 1.1.** *Let  $x \in E^n \subseteq \mathbb{R}^n$  lie in a symmetric convex set with nonempty interior. Assume that  $f, g \in L^+$  are differentiable on  $E$  and  $p + q \neq 0$ .*

- (i) *If  $0 \leq p, q \leq 1$  and  $f, g$  are concave with the same monotonicity, then  $B_n(p, q; f, g, x)$  is Schur concave in  $x$ . If  $f, g$  are decreasing and weakly logarithmically convex, then  $B_n(p, q; f, g, x)$  is Schur convex.*  
(ii) *If  $p, q \geq 1$  and  $f, g$  are convex with opposite monotonicity, then  $B_n(p, q; f, g, x)$  is Schur convex.*  
(iii) *If  $0 \leq p \leq 1, q \geq 1, f$  is decreasing and weakly logarithmically convex, and  $g$  is increasing and convex, then  $B_n(p, q; f, g, x)$  is Schur convex.*

**Theorem 1.2.** *Under analogous assumptions, sufficient conditions are obtained for the Schur convexity and Schur harmonic convexity of  $BH_n(p, q; f, g, x)$ .*

- (i) *Let  $x = (x_1, \dots, x_n) \in E^n \subseteq \mathbb{R}^n$  be in a symmetric convex set with nonempty interior. Let  $f, g \in L^+$ , be differentiable on  $E$ ,  $p \geq 0, q \geq 0$  and  $p + q \neq 0$ . If  $f$  and  $g$  are concave functions, then  $BH_n(p, q; f, g, x)$  is Schur convex with  $x$ .*  
(ii) *Let  $x = (x_1, \dots, x_n) \in E^n \subseteq \mathbb{R}_{++}^n$  be in a symmetric convex set with inner points,  $f, g \in L^+$  and derivable at  $E$ ,  $p \geq 0, q \geq 0$  and  $p + q \neq 0$ . If  $f$  and  $g$  are decreasing and concave functions, then  $BH_n(p, q; f, g, x)$  is Schur harmonically convex with  $x$ .*

## 2. Definitions and lemmas

In order to make the paper self-contained, we recall several basic definitions and lemmas from majorization theory and Schur convexity, which will be used throughout the paper.

**Definition 2.1.** A set  $\Omega \subset \mathbb{R}^n$  is called **symmetric** if  $xP \in \Omega$  for every  $x \in \Omega$  and for every  $n \times n$  permutation matrix  $P$ .

A function  $\varphi : \Omega \rightarrow \mathbb{R}$  is said to be **symmetric** if

$$\varphi(xP) = \varphi(x),$$

for all  $x \in \Omega$  and all permutation matrices  $P$ .

**Definition 2.2** ([11–13]). Let  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  be elements of  $\mathbb{R}^n$ .

(i) The vector  $x$  is said to be **majorized** by  $y$ , written  $x \prec y$ , if

$$\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}, \quad k = 1, 2, \dots, n-1,$$

and

$$\sum_{i=1}^n x_i = \sum_{i=1}^n y_i,$$

where  $x_{[1]} \geq \dots \geq x_{[n]}$  and  $y_{[1]} \geq \dots \geq y_{[n]}$  denote the decreasing rearrangements of  $x$  and  $y$ , respectively.

(ii) A set  $\Omega \subset \mathbb{R}^n$  is **convex** if  $\alpha x + \beta y \in \Omega$  for any  $x, y \in \Omega$ , where  $\alpha, \beta \in [0, 1]$  and  $\alpha + \beta = 1$ .

(iii) A function  $\varphi : \Omega \rightarrow \mathbb{R}$ , defined on a convex set  $\Omega \subset \mathbb{R}^n$ , is called **Schur convex** if

$$x \prec y \implies \varphi(x) \leq \varphi(y).$$

It is called **Schur concave** if  $-\varphi$  is Schur convex.

**Definition 2.3** ([13]). Let  $I \subset \mathbb{R}$  be an interval. A positive function  $\varphi : I \rightarrow \mathbb{R}_{++}$  is said to be **weakly logarithmically convex** if

$$\varphi\left(\frac{x_1 + x_2}{2}\right) \leq \sqrt{\varphi(x_1)\varphi(x_2)}, \quad x_1, x_2 \in I. \tag{2.1}$$

If the inequality is reversed, then  $\varphi$  is said to be **weakly logarithmically concave**.

**Definition 2.4** ([13]). Let  $\Omega \subset \mathbb{R}_+^n$ .

(i) The set  $\Omega$  is called **harmonically convex** if

$$\frac{xy}{\lambda x + (1-\lambda)y} \in \Omega$$

for all  $x, y \in \Omega$  and  $\lambda \in [0, 1]$ , where all operations are understood componentwise.

(ii) A function  $\varphi : \Omega \rightarrow \mathbb{R}_+$  is called **Schur harmonically convex** if

$$x \prec y \implies \varphi(x) \leq \varphi(y).$$

It is called **Schur harmonically concave** if  $-\varphi$  is Schur harmonically convex.

**Lemma 2.5** ([11–13]). Let  $\Omega \subset \mathbb{R}^n$  be a symmetric convex set with nonempty interior, and let  $\varphi : \Omega \rightarrow \mathbb{R}$  be continuous on  $\Omega$  and differentiable in its interior. Then  $\varphi$  is Schur convex (respectively, Schur concave) if and only if  $\varphi$  is symmetric and

$$(x_1 - x_2) \left( \frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2} \right) \geq 0 \quad (\text{respectively, } \leq 0) \tag{2.2}$$

for all  $x = (x_1, \dots, x_n)$  in the interior of  $\Omega$ .

**Lemma 2.6** ([13]). Let  $\varphi : I \rightarrow \mathbb{R}_{++}$  be twice differentiable on an interval  $I \subset \mathbb{R}$ .

(i) The function  $\varphi$  is weakly logarithmically convex on  $I$  if and only if

$$(\varphi'(x))^2 \leq \varphi(x)\varphi''(x), \quad x \in I.$$

(ii) The function  $\varphi$  is weakly logarithmically concave on  $I$  if and only if

$$(\varphi'(x))^2 \geq \varphi(x)\varphi''(x), \quad x \in I.$$

**Lemma 2.7** ([6, 8]). Let  $\Omega \subset \mathbb{R}^n$  be a symmetric harmonically convex set with nonempty interior, and let  $\varphi : \Omega \rightarrow \mathbb{R}$  be continuous on  $\Omega$  and differentiable in its interior. Then  $\varphi$  is Schur harmonically convex (respectively, Schur harmonically concave) if and only if  $\varphi$  is symmetric and

$$(x_1 - x_2) \left( x_1^2 \frac{\partial \varphi}{\partial x_1} - x_2^2 \frac{\partial \varphi}{\partial x_2} \right) \geq 0 \text{ (respectively, } \leq 0) \quad (2.3)$$

for all  $x = (x_1, \dots, x_n)$  in the interior of  $\Omega$ .

**Lemma 2.8** ([11–13]). Let  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ .

(i) The constant vector  $(A_n(x), \dots, A_n(x))$  is majorized by  $x$ ,

$$\underbrace{(A_n(x), \dots, A_n(x))}_n \prec (x_1, \dots, x_n). \quad (2.4)$$

(ii) If  $x_i \neq 0$  for all  $i$ , then the vector  $(H_n(x)^{-1}, \dots, H_n(x)^{-1})$  is majorized by  $(x_1^{-1}, \dots, x_n^{-1})$ ,

$$\left( \underbrace{\frac{1}{H_n(x)}, \dots, \frac{1}{H_n(x)}}_n \right) \prec \left( \frac{1}{x_1}, \dots, \frac{1}{x_n} \right). \quad (2.5)$$

### 3. Proofs of the main theorems

This section contains the detailed proofs of Theorems 1.1 and 1.2, which rely on majorization theory, symmetry, and the differential criteria for Schur convexity presented in Section 2.

#### Proof of Theorem 1.1

*Proof.* Let

$$\bar{B}_n = \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^n f^p(x_i) g^q(x_j),$$

so that  $B_n = \bar{B}_n^{\frac{1}{p+q}}$ . We compute the partial derivatives:

$$\frac{\partial B_n}{\partial x_1} = \frac{\bar{B}_n^{\frac{1}{p+q}-1}}{(p+q)n(n-1)} \times \left[ (f^p(x_1))' \sum_{i=3}^n g^q(x_i) + (f^p(x_1))' g^q(x_2) + (g^q(x_1))' \sum_{i=3}^n f^p(x_i) + (g^q(x_1))' f^p(x_2) \right],$$

$$\frac{\partial B_n}{\partial x_2} = \frac{\bar{B}_n^{\frac{1}{p+q}-1}}{(p+q)n(n-1)} \times \left[ (f^p(x_2))' \sum_{i=3}^n g^q(x_i) + (f^p(x_2))' g^q(x_1) + (g^q(x_2))' \sum_{i=3}^n f^p(x_i) + (g^q(x_2))' f^p(x_1) \right].$$

Obviously,  $B_n(p, q; f, g, x)$  is symmetric in  $x_1, \dots, x_n$ , without loss of generality, we may assume that  $x_1 \geq x_2$ . Then

$$\begin{aligned} \Delta_1 &:= (x_1 - x_2) \left( \frac{\partial B_n}{\partial x_1} - \frac{\partial B_n}{\partial x_2} \right) \\ &= \frac{(x_1 - x_2) \bar{B}_n^{\frac{1}{p+q}-1}}{(p+q)n(n-1)} \{ [(f^p(x_1))' - (f^p(x_2))'] \sum_{i=3}^n g^q(x_i) + (f^p(x_1))' g^q(x_2) \\ &\quad - (f^p(x_2))' g^q(x_1) + [(g^q(x_1))' - (g^q(x_2))'] \sum_{i=3}^n f^p(x_i) + (g^q(x_1))' f^p(x_2) \\ &\quad - (g^q(x_2))' f^p(x_1) \}. \end{aligned}$$

(i) For  $0 \leq p, q \leq 1$ , if  $f$  and  $g$  are concave functions and have the same monotonicity, because

$$(f^p(x))' = p f^{p-1}(x) f'(x),$$

$$(f^p(x))'' = p f^{p-2}(x) [(p-1)(f'(x))^2 + f(x) f''(x)],$$

then  $f^p$  and  $g^q$  are concave functions and have the same monotonicity.

If  $f, g$  are concave functions and increasing, then

$$\begin{aligned} (f^p(x_1))' g^q(x_2) - (f^p(x_2))' g^q(x_1) &\leq (f^p(x_1))' g^q(x_1) - (f^p(x_2))' g^q(x_1) \\ &= g^q(x_1) [(f^p(x_1))' - (f^p(x_2))'] \leq 0. \end{aligned}$$

By the same arguments, it follows that

$$(g^q(x_1))' f^p(x_2) - (g^q(x_2))' f^p(x_1) \leq 0.$$

It is easy to know that

$$[(f^p(x_1))' - (f^p(x_2))'] \sum_{i=3}^n g^q(x_i) \leq 0, [(g^q(x_1))' - (g^q(x_2))'] \sum_{i=3}^n f^p(x_i) \leq 0,$$

so  $\Delta_1 \leq 0$ .

If  $f, g$  are concave functions and decreasing, then

$$\begin{aligned} (g^q(x_1))' f^p(x_2) - (g^q(x_2))' f^p(x_1) &\leq (g^q(x_2))' f^p(x_2) - (g^q(x_2))' f^p(x_1) \\ &= (g^q(x_2))' (f^p(x_2) - f^p(x_1)) \leq 0. \end{aligned}$$

By the same arguments, it follows that

$$(f^p(x_1))' g^q(x_2) - (f^p(x_2))' g^q(x_1) \leq 0,$$

so  $\Delta_1 \leq 0$ .

By Lemma 2.5,  $B_n(p, q; f, g, x)$  is Schur concave in  $x$ .

If  $f$  and  $g$  are decreasing and weakly logarithmically convex functions, by Lemma 2.6, it follows that

$$\begin{aligned} (f^p(x))'' &= pf^{p-2}(x)[(p-1)(f'(x))^2 + f(x)f''(x)] \\ &= pf^{p-2}(x)[p(f'(x))^2 + f(x)f''(x) - (f'(x))^2] \geq 0, \end{aligned}$$

so  $f^p$  is convex function. By the same arguments,  $g^q$  is also convex function. And then

$$\begin{aligned} (g^q(x_1))'f^p(x_2) - (g^q(x_2))'f^p(x_1) &\geq (g^q(x_2))'f^p(x_2) - (g^q(x_2))'f^p(x_1) \\ &= (g^q(x_2))'(f^p(x_2) - f^p(x_1)) \geq 0. \end{aligned}$$

By the same arguments, it follows that

$$(f^p(x_1))'g^q(x_2) - (f^p(x_2))'g^q(x_1) \geq 0,$$

It is easy to know that

$$[(f^p(x_1))' - (f^p(x_2))'] \sum_{i=3}^n g^q(x_i) \geq 0, [(g^q(x_1))' - (g^q(x_2))'] \sum_{i=3}^n f^p(x_i) \geq 0,$$

so  $\Delta_1 \geq 0$ , by Lemma 2.5, it follows that  $B_n(p, q; f, g, x)$  is Schur convex with  $x$ .

(ii) For  $p, q \geq 1$ , if  $f, g$  are convex functions with opposite monotonicity, then  $f^p, g^q$  are convex and have the same monotonicity as  $f, g$ , respectively, without loss of generality, we may assume that  $f$  is decreasing,  $g$  is increasing, then

$$\begin{aligned} (g^q(x_1))'f^p(x_2) - (g^q(x_2))'f^p(x_1) &\geq (g^q(x_1))'f^p(x_1) - (g^q(x_2))'f^p(x_1) \\ &= f^p(x_1)[(g^q(x_1))' - (g^q(x_2))'] \geq 0, \\ (f^p(x_1))'g^q(x_2) - (f^p(x_2))'g^q(x_1) &\geq (f^p(x_1))'g^q(x_1) - (f^p(x_2))'g^q(x_1) \\ &= g^q(x_1)[(f^p(x_1))' - (f^p(x_2))'] \geq 0. \end{aligned}$$

It is easy to know that

$$[(f^p(x_1))' - (f^p(x_2))'] \sum_{i=3}^n g^q(x_i) \geq 0, [(g^q(x_1))' - (g^q(x_2))'] \sum_{i=3}^n f^p(x_i) \geq 0,$$

hence  $\Delta_1 \geq 0$ . By Lemma 2.5,  $B_n(p, q; f, g, x)$  is Schur convex in  $x$ .

(iii) For  $0 \leq p \leq 1, q \geq 1$ , if  $f(x)$  is decreasing and weakly logarithmically convex function,  $g(x)$  is increasing and convex function, because  $(f^p(x))' = pf^{p-1}(x)f'(x) \leq 0$ , then  $f^p$  and  $f$  has the same monotonicity.

For  $0 \leq p \leq 1$ , by Lemma 2.6 we have

$$\begin{aligned} (f^p(x))'' &= pf^{p-2}(x)[f(x)f''(x) - (1-p)(f'(x))^2] \\ &\geq pf^{p-2}(x)[f(x)f''(x) - (f'(x))^2] \geq 0, \end{aligned}$$

so  $f^p$  is convex function. For  $q \geq 1$ , it is easy to know that  $g^q$  is increasing convex function. Hence

$$\begin{aligned} (g^q(x_1))'f^p(x_2) - (g^q(x_2))'f^p(x_1) &\geq (g^q(x_1))'f^p(x_1) - (g^q(x_2))'f^p(x_1) \\ &= f^p(x_1)[(g^q(x_1))' - (g^q(x_2))'] \geq 0, \\ (f^p(x_1))'g^q(x_2) - (f^p(x_2))'g^q(x_1) &\geq (f^p(x_1))'g^q(x_1) - (f^p(x_2))'g^q(x_1) \\ &= g^q(x_1)[(f^p(x_1))' - (f^p(x_2))'] \geq 0, \end{aligned}$$

$$[(f^p(x_1))' - (f^p(x_2))'] \sum_{i=3}^n g^q(x_i) \geq 0, [(g^q(x_1))' - (g^q(x_2))'] \sum_{i=3}^n f^p(x_i) \geq 0,$$

so  $\Delta_1 \geq 0$ , by Lemma 2.5, it follows that  $B_n(p, q; f, g, x)$  is Schur convex with  $x$ .

This completes the proof of Theorem 1.1. □

**Proof of Theorem 1.2**

*Proof.* Write

$$\bar{F} = \frac{1}{n^2} \sum_{i,j=1}^n \frac{1}{f^p(x_i)g^q(x_j)}.$$

By computing,

$$\begin{aligned} \frac{\partial BH_n}{\partial x_1} &= -\frac{1}{p+q} \bar{F}^{-\frac{1}{p+q}-1} \frac{\partial \bar{F}}{\partial x_1} \\ &= -\frac{1}{n^2(p+q)} \bar{F}^{-\frac{1}{p+q}-1} \left[ \sum_{i=1}^n (f^p(x_i)g^q(x_i))^{-2} (pf'(x_i)f^{p-1}(x_i)g^q(x_i)) \right. \\ &\quad \left. + \sum_{i=1}^n (f^p(x_i)g^q(x_i))^{-2} (qg'(x_i)g^{q-1}(x_i)f^p(x_i)) \right], \end{aligned}$$

and

$$\begin{aligned} \frac{\partial BH_n}{\partial x_2} &= -\frac{1}{p+q} \bar{F}^{-\frac{1}{p+q}-1} \frac{\partial \bar{F}}{\partial x_2} \\ &= -\frac{1}{n^2(p+q)} \bar{F}^{-\frac{1}{p+q}-1} \left[ \sum_{i=1}^n (f^p(x_i)g^q(x_i))^{-2} (pf'(x_i)f^{p-1}(x_i)g^q(x_i)) \right. \\ &\quad \left. + \sum_{i=1}^n (f^p(x_i)g^q(x_i))^{-2} (qg'(x_i)g^{q-1}(x_i)f^p(x_i)) \right]. \end{aligned}$$

Obviously,  $BH_n(p, q; f, g, x)$  is symmetric in  $x_1, \dots, x_n$ . Without loss of generality, we may assume that  $x_1 \geq x_2$ .

(i) Notice that when  $k \geq 0$ ,  $h(x) > 0$  and  $h''(x) \leq 0$ , then the function  $m(x) = h^{-k-1}(x)h'(x)$  is decreasing, we have

$$\begin{aligned} \Delta_2 &:= (x_1 - x_2) \left( \frac{\partial BH_n}{\partial x_1} - \frac{\partial BH_n}{\partial x_2} \right) \\ &= -(x_1 - x_2) \frac{1}{n^2(p+q)} \bar{F}^{-\frac{1}{p+q}-1} \\ &\quad \times \left\{ \sum_{i=1}^n [(f^p(x_i)g^q(x_i))^{-2} (pf'(x_i)f^{p-1}(x_i)g^q(x_i)) \right. \\ &\quad \left. - (f^p(x_2)g^q(x_i))^{-2} (pf'(x_2)f^{p-1}(x_2)g^q(x_i))] \right. \\ &\quad \left. + \sum_{i=1}^n [(g^q(x_1)f^p(x_i))^{-2} (qg'(x_1)g^{q-1}(x_1)f^p(x_i)) \right. \\ &\quad \left. - (g^q(x_2)f^p(x_i))^{-2} (qg'(x_2)g^{q-1}(x_2)f^p(x_i))] \right\} \\ &= -(x_1 - x_2) \frac{1}{n^2(p+q)} \bar{F}^{-\frac{1}{p+q}-1} \\ &\quad \times \left[ \sum_{i=1}^n pg^{-q}(x_i) (f^{-p-1}(x_1)f'(x_1) - f^{-p-1}(x_2)f'(x_2)) \right. \\ &\quad \left. + \sum_{i=1}^n qf^{-p}(x_i) (g^{-q-1}(x_1)g'(x_1) - g^{-q-1}(x_2)g'(x_2)) \right] \geq 0. \end{aligned}$$

By Lemma 2.5, it follows that  $BH_n(p, q; f, g, x)$  is Schur convex with  $x$ .

(ii) Notice that when  $x > 0$ ,  $k \geq 0$  and  $h(x) > 0$ ,  $h'(x) \leq 0$ ,  $h''(x) \leq 0$ , then the function  $n(x) = x^2 h^{-k-1}(x) h'(x)$  is decreasing, we have

$$\begin{aligned} \Delta_3 &:= (x_1 - x_2) \left( x_1^2 \frac{\partial BH_n}{\partial x_1} - x_2^2 \frac{\partial BH_n}{\partial x_2} \right) \\ &= - (x_1 - x_2) \frac{1}{n^2(p+q)} \bar{F}^{-\frac{1}{p+q}-1} \\ &\quad \times \left\{ \sum_{i=1}^n [x_1^2 (f^p(x_1) g^q(x_i))^{-2} (p f'(x_1) f^{p-1}(x_1) g^q(x_i)) \right. \\ &\quad \left. - x_2^2 (f^p(x_2) g^q(x_i))^{-2} (p f'(x_2) f^{p-1}(x_2) g^q(x_i))] \right. \\ &\quad \left. + \sum_{i=1}^n [x_1^2 (g^q(x_1) f^p(x_i))^{-2} (q g'(x_1) g^{q-1}(x_1) f^p(x_i)) \right. \\ &\quad \left. - x_2^2 (g^q(x_2) f^p(x_i))^{-2} (q g'(x_2) g^{q-1}(x_2) f^p(x_i))] \right\} \\ &= - (x_1 - x_2) \frac{1}{n^2(p+q)} \bar{F}^{-\frac{1}{p+q}-1} \\ &\quad \times \left[ \sum_{i=1}^n p g^{-q}(x_i) (x_1^2 f^{-p-1}(x_1) f'(x_1) - x_2^2 f^{-p-1}(x_2) f'(x_2)) \right. \\ &\quad \left. + \left[ \sum_{i=1}^n q f^{-p}(x_i) (x_1^2 g^{-q-1}(x_1) g'(x_1) - x_2^2 g^{-q-1}(x_2) g'(x_2)) \right] \right] \geq 0. \end{aligned}$$

By Lemma 2.7, it follows that  $BH_n(p, q; f, g, x)$  is Schur harmonically convex with  $x$ .

The proof of Theorem 1.2 is complete. □

#### 4. Applications

We derive several inequalities as consequences of the Schur convexity results, including:

- Upper and lower bounds involving arithmetic and harmonic means;
- Integral inequalities for decreasing integrable functions;
- Comparison inequalities between Bonferroni means and generalized harmonic Bonferroni means.

These results generalize many known inequalities in the literature.

**Corollary 4.1.** *Let  $x = (x_1, \dots, x_n) \in E^n \subseteq \mathbb{R}^n$ ,  $f, g \in L^+$ ,  $0 \leq p, q \leq 1$  and  $p + q \neq 0$ .*

(i) *If  $f, g$  are concave functions and have the same monotonicity, then*

$$B_n(p, q; f, g, x) \leq [f(A_n(x))]^{\frac{p}{p+q}} [g(A_n(x))]^{\frac{q}{p+q}}. \tag{4.1}$$

(ii) *If  $f, g$  are weak logarithmically convex functions and decreasing, then*

$$B_n(p, q; f, g, x) \geq [f(A_n(x))]^{\frac{p}{p+q}} [g(A_n(x))]^{\frac{q}{p+q}}. \tag{4.2}$$

*Proof.* By Theorem 1.1(i) and Definition 2.2, from

$$\underbrace{(A_n(x), \dots, A_n(x))}_n \prec (x_1, \dots, x_n),$$

it follows that inequality (4.1), (4.2) hold.

The proof of Corollary 4.1 is complete. □

In particular, let  $p = q = 1$ ,  $f(x) = g(x)$  is monotone concave function, then

$$\frac{1}{n(n-1)} \sum_{i,j=1;i \neq j}^n f(x_i)f(x_j) \leq [f(A_n(x))]^2. \tag{4.3}$$

This yields an upper bound inequality for the Bonferroni mean.

**Corollary 4.2.** *Let  $x = (x_1, \dots, x_n) \in \mathbb{R}_{++}^n$ . If  $0 \leq p, q \leq 1$ , and  $p + q \neq 0$ , then*

$$B_n^{p,q}(x) \leq A_n(x). \tag{4.4}$$

*Proof.* Let  $I(x) = f(x) = g(x) = x$ . Then

$$B_n(p, q; I, I, x) = \left( \frac{1}{n(n-1)} \sum_{i,j=1;i \neq j}^n x_i^p x_j^q \right)^{\frac{1}{p+q}} = B_n^{p,q}(x),$$

by Corollary 4.1(i), it follows that the inequality (4.4) holds.

The proof of Corollary 4.2 is complete. □

**Corollary 4.3.** *Let  $x = (x_1, \dots, x_n) \in \mathbb{R}_{++}^n$ . If  $0 \leq p, q \leq 1$  and  $p + q \neq 0$ ,  $\alpha, \beta < 0$ , then*

$$B_n(p, q; x^\alpha, x^\beta, x) \geq [A_n(x)]^{\frac{\alpha p + \beta q}{p+q}}. \tag{4.5}$$

*Proof.* Let  $f(x) = x^\alpha$ ,  $g(x) = x^\beta$ . It is easy to verify that  $f$  and  $g$  are decreasing and weakly logarithmically convex, by Corollary 4.1(ii), it follows that the inequality (4.5) holds.

The proof of Corollary 4.3 is complete. □

**Corollary 4.4.** *Let  $x = (x_1, \dots, x_n) \in E^n \subseteq \mathbb{R}^n$ ,  $f, g \in L^+$ . If  $0 \leq p \leq 1$ ,  $q \geq 1$ ,  $f$  is a decreasing and weakly logarithmically convex function,  $g$  is increasing and convex function, or  $p, q \geq 1$ ,  $f, g$  are convex functions with opposite monotonicity, then*

$$B_n(p, q; f, g, x) \geq [f(A_n(x))]^{\frac{p}{p+q}} [g(A_n(x))]^{\frac{q}{p+q}}. \tag{4.6}$$

Special, let  $p = q = 1$ . If  $f, g$  are convex functions with opposite monotonicity, by corollary 4.4, we have

$$\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^n f(x_i)g(x_j) \geq f(A_n(x))g(A_n(x)). \tag{4.7}$$

Let  $f(x) = \frac{1}{x}$ ,  $g(x) = x$ , ( $x > 0$ ). By Corollary 4.4, we have

**Corollary 4.5.** *Let  $x = (x_1, \dots, x_n) \in \mathbb{R}_{++}^n$ . If  $0 \leq p \leq 1$ ,  $q \geq 1$  and  $\sum_{i=1}^n x_i = 1$ , then*

$$\sum_{\substack{i,j=1 \\ i \neq j}}^n \frac{x_j^q}{x_i^p} \geq n^{p-q+1}(n-1). \tag{4.8}$$

**Corollary 4.6.** *Let  $x = (x_1, \dots, x_n) \in \mathbb{R}_{++}^n$ . If  $f, g$  are nonnegative, integrable, and decreasing functions on  $(0, +\infty)$ ,  $0 \leq p, q \leq 1$ , then*

$$\begin{aligned} & \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^n \left( \int_0^{x_i} f(t)dt \right)^p \left( \int_0^{x_j} g(t)dt \right)^q \\ & \leq \left( \int_0^{A_n(x)} f(t)dt \right)^p \left( \int_0^{A_n(x)} g(t)dt \right)^q. \end{aligned} \tag{4.9}$$

*Proof.* Let

$$F(x) = \int_0^x f(t)dt, \quad G(x) = \int_0^x g(t)dt.$$

Then  $F, G$  are increasing, and  $F''(x) = f'(x) \leq 0, G''(x) = g'(x) \leq 0$ , by Corollary 4.1, it follows that the inequality (4.9) holds.

The proof of Corollary 4.6 is complete. □

*Remark 4.7.* (i) Let  $x = (x_1, \dots, x_n) \in \mathbb{R}_{++}^n$ . If  $0 \leq p, q \leq 1$  and  $p + q = 1$ ,  $f$  be nonnegative integrable decreasing function on  $(0, +\infty)$ , by Corollary 4.6, we have

$$\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^n \left( \int_0^{x_i} f(t)dt \right)^p \left( \int_0^{x_j} f(t)dt \right)^q \leq \int_0^{A_n(x)} f(t)dt. \tag{4.10}$$

Let  $p = q = \frac{1}{2}, n = 2, a, b > 0$ , we have

$$\int_0^a f(t)dt \int_0^b f(t)dt \leq \left( \int_0^{\frac{a+b}{2}} f(t)dt \right)^2. \tag{4.11}$$

Let  $f(x)$  be a nonnegative, geometrically convex (for the definition, see [14]) and decreasing function on  $(0, +\infty)$ ,  $a, b > 0$ . Combining results from [14, 15] with inequality (4.11), we obtain the bilateral inequality:

$$\left( \int_0^{\sqrt{ab}} f(t)dt \right)^2 \leq \int_0^a f(t)dt \int_0^b f(t)dt \leq \left( \int_0^{\frac{a+b}{2}} f(t)dt \right)^2. \tag{4.12}$$

(ii) Let  $x = (x_1, \dots, x_n) \in \mathbb{R}_{++}^n$ . If  $f, g$  are nonnegative integrable decreasing functions on  $(0, +\infty)$ , by Corollary 4.6 and Chebyshevs inequality, we have

$$\sum_{\substack{i,j=1 \\ i \neq j}}^n \int_0^{x_i} f(t)dt \int_0^{x_j} g(t)dt \leq (n-1) \int_0^{A_n(x)} f(t)g(t)dt \sum_{i=1}^n x_i. \tag{4.13}$$

**Corollary 4.8.** (i) Let  $x = (x_1, \dots, x_n) \in E^n \subseteq \mathbb{R}^n, p, q \geq 0$  and  $p + q \neq 0, f, g \in L^+$ . If  $f, g$  is concave function, then

$$BH_n(p, q; f, g, x) \geq [f(A_n(x))]^{\frac{p}{p+q}} [g(A_n(x))]^{\frac{q}{p+q}}. \tag{4.14}$$

(ii) Let  $x = (x_1, \dots, x_n) \in E^n \subseteq \mathbb{R}_{++}^n, p, q \geq 0$  and  $p + q \neq 0, f, g \in L^+$ . If  $f, g$  is decreasing concave function, then

$$BH_n(p, q; f, g, x) \geq [f(H_n(x))]^{\frac{p}{p+q}} [g(H_n(x))]^{\frac{q}{p+q}}. \tag{4.15}$$

*Proof.* (i) By Theorem 1.2(i)and Definition 2.2, from

$$\underbrace{(A_n(x), \dots, A_n(x))}_n \prec (x_1, \dots, x_n),$$

we have

$$BH_n(p, q; f, g, x) \geq [f(A_n(x))]^{\frac{p}{p+q}} [g(A_n(x))]^{\frac{q}{p+q}}.$$

(ii) By Theorem 1.2(ii), Definition 2.4, from

$$\left( \underbrace{\left( \frac{1}{H_n(x)}, \dots, \frac{1}{H_n(x)} \right)}_n \right) \prec \left( \frac{1}{x_1}, \dots, \frac{1}{x_n} \right),$$

we have

$$BH_n(p, q; f, g, x) \geq [f(H_n(x))]^{\frac{p}{p+q}} [g(H_n(x))]^{\frac{q}{p+q}}.$$

The proof of Corollary 4.8 is complete. □

Let  $f(x) = g(x) = x$ , by Corollary 4.6, we get the lower bound inequality of generalized Bonferroni harmonically mean.

**Corollary 4.9.** *Let  $x = (x_1, \dots, x_n) \in \mathbb{R}_{++}^n$ ,  $p, q \geq 0$  and  $p + q \neq 0$ . Then*

$$BH_n(p, q; x) \geq A_n(x). \tag{4.16}$$

From Corollary 4.1(i) and Corollary 4.8(i) we obtain the following result.

**Corollary 4.10.** *Let  $x = (x_1, \dots, x_n) \in E^n \subseteq \mathbb{R}^n$ ,  $0 \leq p, q \leq 1$  and  $p + q \neq 0$ . If  $f(x) > 0$ ,  $g(x) > 0$  are concave functions and have the same monotonicity, then*

$$B_n(p, q; f, g, x) \leq [f(A_n(x))]^{\frac{p}{p+q}} [g(A_n(x))]^{\frac{q}{p+q}} \leq BH_n(p, q; f, g, x). \tag{4.17}$$

In particular, let  $f(x) = g(x) = x > 0$ . For  $0 \leq p, q \leq 1$  and  $p + q \neq 0$ , we have

$$B_n^{p,q}(x) \leq A_n(x) \leq BH_n(p, q; x). \tag{4.18}$$

### 5. Norm inequalities of Bonferroni functional mean

We introduce operator norms associated with functional Bonferroni means and establish sharp bounds in terms of the norms of the underlying linear functionals. These results provide a functional-analytic interpretation of Bonferroni-type inequalities.

Let  $E \subseteq \mathbb{R}$  be a closed, symmetric, convex sets,  $x \in E$ , and let  $\|\cdot\|$  be a norm on  $E$ ,  $f(x)$  be a bounded linear functional on  $E \rightarrow (0, +\infty)$ ,  $A_n(x) = \frac{1}{n} \sum_{i=1}^n x_i$ ,  $x = (x_1, \dots, x_n) \in E^n$ , we define the following norm:

$$\begin{aligned} \|x\| &= \max_{x_i \in E} \{|x_i|, i = 1, \dots, n\}, \\ \|f\| &= \sup_{x \in E, x \neq 0} \frac{f(x)}{\|x\|}, \\ \|A_n\| &= \sup_{x \in E^n, x \neq 0} \frac{\|A_n(x)\|}{\|x\|}, \\ \|B_n(f, g)\|_{p,q} &= \sup_{x \in E^n, x \neq 0} \frac{B_n(p, q; f, g, x)}{\|x\|}. \end{aligned}$$

Obviously  $A_n(x)$  is bounded linear operator on  $E^n$  and  $\|A_n\| \leq 1$ . We have the following conclusions

**Theorem 5.1.** *Let  $f, g$  be positive, bounded linear functionals on  $E$ ,  $p, q \geq 0$  and  $p + q \neq 0$ ,  $x = (x_1, \dots, x_n) \in E^n$ . Then*

$$\|B_n(f, g)\|_{p,q} \leq (\|f\|^p \|g\|^q)^{\frac{1}{p+q}}. \tag{5.1}$$

*Proof.*

$$\begin{aligned} \|B_n(f, g)\|_{p,q} &= \sup_{x \in E^n, x \neq 0} \frac{B_n(p, q; f, g, x)}{\|x\|} \\ &= \sup_{x \in E^n, x \neq 0} \frac{\left(\frac{1}{n(n-1)} \sum_{i,j=1; i \neq j}^n f^p(x_i) \cdot g^q(x_j)\right)^{\frac{1}{p+q}}}{\|x\|} \\ &\leq \frac{\left(\frac{1}{n(n-1)} \sum \|f\|^p \|x_i\|^p \|g\|^q \|x_j\|^q\right)^{\frac{1}{p+q}}}{\|x\|} \end{aligned}$$

$$\begin{aligned} &\leq \frac{\left(\frac{1}{n(n-1)} \sum \|f\|^p \|g\|^q\right)^{\frac{1}{p+q}} \|x\|}{\|x\|} \\ &= (\|f\|^p \|g\|^q)^{\frac{1}{p+q}}. \end{aligned}$$

The proof of Theorem 5.1 is complete. □

**Theorem 5.2.** *Let  $f, g$  be two positive bounded linear functions on  $E \subset \mathbb{R}$ .  $f, g$  are concave and have the same monotonicity,  $0 \leq p, q \leq 1$ ,  $x = (x_1, \dots, x_n) \in E^n \subset \mathbb{R}^n$ . Then*

$$\|B_n(f, g)\|_{p,q} \leq (\|f\|^p \|g\|^q)^{\frac{1}{p+q}} \|A_n\| \leq (\|f\|^p \|g\|^q)^{\frac{1}{p+q}}. \tag{5.2}$$

*Proof.* By Corollary 4.1 we have

$$\begin{aligned} \|B_n(f, g)\|_{p,q} &= \sup_{x \in E^n, x \neq 0} \frac{B_n(p, q; f, g, x)}{\|x\|} \\ &\leq \sup_{x \in E^n, x \neq 0} \frac{[f(A_n(x))]^{\frac{p}{p+q}} [g(A_n(x))]^{\frac{q}{p+q}}}{\|x\|} \\ &\leq \frac{(\|f\| \|A_n(x)\|)^{\frac{p}{p+q}} (\|g\| \|A_n(x)\|)^{\frac{q}{p+q}}}{\|x\|} \\ &= (\|f\|^p \|g\|^q)^{\frac{1}{p+q}} \frac{\|A_n(x)\|}{\|x\|} \\ &\leq (\|f\|^p \|g\|^q)^{\frac{1}{p+q}} \frac{\|A_n\| \|x\|}{\|x\|} \\ &\leq (\|f\|^p \|g\|^q)^{\frac{1}{p+q}}. \end{aligned}$$

The proof of Theorem 5.2 is complete. □

In Theorem 5.2, let  $p + q = 1$ . For any  $x = (x_1, \dots, x_n) \in E^n \subset \mathbb{R}^n$ , we have

$$\sum_{i,j=1, i \neq j}^n f^p(x_i) g^q(x_j) \leq n(n-1) \|f\|^p \|g\|^q \|A_n(x)\| \leq n(n-1) \|f\|^p \|g\|^q \|x\|. \tag{5.3}$$

For Bonferroni mean:  $B_n^{p,q}(x)$ , let  $0 \leq p, q \leq 1$ . By Theorem 5.2 we have

$$\|B_n^{p,q}\| \leq \|A_n\| \leq 1. \tag{5.4}$$

Let  $x(t) \in C_{[a,b]}$ ,  $x(t) = (x_1(t), \dots, x_n(t)) \in C_{[a,b]}^n$ ,  $v(t) \in C_{[a,b]}$  is bounded variation function. We define

$$\begin{aligned} f(x) &= \int_a^b x(t) dv(t), \\ \|x(t)\| &= \max_{t \in [a,b]} |x(t)| \end{aligned}$$

and

$$\|x(t)\| = \max\{\|x_i(t)\|, i = 1, \dots, n\}.$$

By F.Riesz Theorem we know that  $f$  is a linear continuous functional and  $\|f\| \leq \int_a^b v$  (see [16]). By Theorem 5.1, we have

**Corollary 5.3.** Let  $x(t) = (x_1(t), \dots, x_n(t)) \in C_{[a,b]}^n$ , where  $v(t) \in C_{[a,b]}$  is a function of bounded variation and  $x_i(t)v'(t) > 0, i = 1, \dots, n$ . If  $p, q \geq 0$  and  $p + q \neq 0$ , then

$$\|B_n(f, f)\|_{p,q} \leq \bigvee_a^b(v). \quad (5.5)$$

## 6. Conclusion

In this paper, we introduced functional Bonferroni means and functional generalized Bonferroni harmonic means, extending the classical Bonferroni framework to a functional setting. Using majorization theory and differential criteria for Schur convexity, we established sufficient conditions for Schur convexity, Schur concavity, and Schur harmonic convexity of these means. A variety of applications were obtained, including separation inequalities, integral inequalities, and norm inequalities.

The results presented here unify and generalize several known inequalities for classical means. Future work may focus on discrete analogues, probabilistic interpretations, or extensions to other classes of generalized means.

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## Use of AI Tools

AI tools were not employed in generating, analyzing, or interpreting the results.

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