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Research Article

Exploring Hermite–Hadamard-type Inequalities via ψ-conformable Fractional Integral Operators

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Abstract

This study presents novel Hermite–Hadamard inequalities for convex functions using ψ-conformable fractional integral operators. These operators are extensions of several significant fractional operators, including the Riemann-Liouville and Hadamard operators. Furthermore, we present generalized midpoint- and trapezoidal-type inequalities for these fractional integrals, which are extensions of previous studies.

Keywords: Hermite–Hadamard-type inequalities, ψ-conformable, convex function

2020 MSC: 26D10, 26A33, 26A51, 26D15.

1. Introduction

Convex function analysis is a deep generalization model for many applications that starts with real-valued functions of a real variable. Convexity theory offers a cohesive framework for creating extremely efficient and robust numerical tools to address issues across many mathematical fields. Various intriguing generalizations and extensions of multiple forms of convexity have been utilized in optimization and mathematical inequalities, especially Hermite–Hadamard-type inequalities, as referenced in [7]-[4]. In [2], the authors proved the following generalized Hermite–Hadamard-type inequalities.
Let $\psi : [a,b] \to \mathbf{R}$ be a monotone increasing function such that the derivative $\psi' > 0$ is continuous on (a,b) and $\beta > 0$. If g is a convex function on $[a,b]$, then

$$g\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\beta+1)}{2A_\psi(1)}\left[{}^{\psi}\mathscr{I}^{\beta}_{(\frac{a+b}{2})^-}G(a) + {}^{\psi}\mathscr{I}^{\beta}_{(\frac{a+b}{2})^+}G(b)\right] \le \frac{g(a)+g(b)}{2}, \tag{1.1}$$

where

$$ {}^{\psi}\mathscr{I}^{\beta}_{a^+}g(x) = \frac{1}{\Gamma(\beta)}\int_a^x \psi'(t)\left[\psi(x)-\psi(t)\right]^{\beta-1} g(t)\,dt,$$

$$ {}^{\psi}\mathscr{I}^{\beta}_{b^-}g(x) = \frac{1}{\Gamma(\beta)}\int_x^b \psi'(t)\left[\psi(t)-\psi(x)\right]^{\beta-1} g(t)\,dt,$$

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and

$$G(s) = g(s) + g(a+b-s),$$

$$A_{\psi}(1) = \left(\psi(b) - \psi\left(\frac{a+b}{2}\right)\right)^{\beta} + \left(\psi\left(\frac{a+b}{2}\right) - \psi(a)\right)^{\beta}.$$

If $|g'|$ is a convex mapping on $[a,b]$, then the generalized trapezoid-type inequality is given as [2, Theorem 4.4].

$$\left|\frac{g(a)+g(b)}{2} - \frac{\Gamma(\beta+1)}{2A_{(\psi)}(1)}\left[{}^{\psi}\mathscr{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{-}}G(a) + {}^{\psi}\mathscr{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}G(b)\right]\right|$$

$$\leq \frac{b-a}{4A_{\psi}(1)}\left[|g'(a)| + |g'(b)|\right]\int_0^1 (A_{\psi}(1) - A_{\psi}(s))\,ds.$$

where

$$A_{\psi}(s) = \left(\psi(b) - \psi\left(\frac{s}{2}a + \frac{2-s}{2}b\right)\right)^{\beta} + \left(\psi\left(\frac{2-s}{2}a + \frac{s}{2}b\right) - \psi(a)\right)^{\beta}.$$

The generalized midpoint-type inequality is formatted as follows [2, Theorem 3.4]:

$$\left|\frac{\Gamma(\beta+1)}{2A_{\psi}(1)}\left[{}^{\psi}\mathscr{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{-}}G(a) + {}^{\psi}\mathscr{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}G(b)\right] - g\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{b-a}{4A_{\psi}(1)}\left[|g'(a)| + |g'(b)|\right]\int_0^1 A_{\psi}(s)ds.$$

In the recent paper [1], the authors introduced a new operator called generalized ψ-conformable fractional integrals, which are defined as follows:

Definition 1.1. Let $0 \leq a < b < \infty$, $f : [a,b] \subset [0,+\infty) \to \mathbb{R}$ be a continuous function and $\psi : [a,b] \to \mathbf{R}$ be a monotone increasing function such that the derivative $\psi' > 0$ is continuous on (a,b). The generalized left and right ψ-conformable fractional integral operators of the function f, with order $\beta > 0$, are defined as

$$^{\psi}_{\mathscr{CF}}I^{\beta}_{a^+}f(x) = \frac{1}{\Gamma(\beta)}\int_a^x \psi'(t)\left[\frac{\psi^{\alpha}(x) - \psi^{\alpha}(t)}{\alpha}\right]^{\beta-1} f(t)\,d_{\psi}t, \tag{1.2}$$

and

$$^{\psi}_{\mathscr{CF}}I^{\beta}_{b^-}f(x) = \frac{1}{\Gamma(\beta)}\int_x^b \psi'(t)\left[\frac{\psi^{\alpha}(t) - \psi^{\alpha}(x)}{\alpha}\right]^{\beta-1} f(t)\,d_{\psi}t, \tag{1.3}$$

where

$$d_{\psi}t = \frac{dt}{\psi^{1-\alpha}(t)}.$$

The results are already known for the special choices of ψ, α and β.

1. By setting $\alpha = 1$, the operators simplify to the ψ-Hilfer integral operators of order $\beta > 0$.
2. For $\psi(t) = t$, the Katugompola operators of order $\beta > 0$ and parameter $\alpha > 0$ are obtained.
3. When $\psi(t) = t$, $\alpha = 1$, the operators are reduced to Riemann-Liouville integral operators.
4. By selecting $\psi(t) = t$, $\alpha = 1$, and $\beta = 1$, the operators are reduced to classical Riemann integrals.
5. The α-Hadamard operators of order $\beta > 0$ are obtained by setting $\psi(t) = \ln(t)$ and $a > 1$. These operators are denoted in (1.4) and (1.5).
6. By defining $\psi(t) = \ln(t)$, $\alpha = 1$, and $a > 1$, we obtain Hadamard operators of order $\beta > 0$.

Definition 1.2. Let $1 \leq a < b < \infty$, $f : [a,b] \subset [0,+\infty) \to \mathbb{R}$ be a continuous function. The left and right α-Hadamard integral operators with order $\beta > 0$, are defined as

$$^{\alpha}\mathscr{H}^{\beta}_{a^+} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left[\frac{\ln^{\alpha}(x) - \ln^{\alpha}(t)}{\alpha}\right]^{\beta-1} \frac{f(t)}{t} \frac{dt}{\ln^{1-\alpha}(t)}, \tag{1.4}$$

and

$$^{\alpha}\mathscr{H}^{\beta}_{b^-} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b \left[\frac{\ln^{\alpha}(t) - \ln^{\alpha}(x)}{\alpha}\right]^{\beta-1} \frac{f(t)}{t} \frac{dt}{\ln^{1-\alpha}(t)}. \tag{1.5}$$

The objective of this study is to generalize the Hermite–Hadamard inequalities previously established using ψ-conformable fractional integral operators.

2. Hermite–Hadamard inequalities

This is the first result about Hermite–Hadamard-type inequalities for convex functions with general ψ-conformable fractional integral operators.

Theorem 2.1. *Let $0 \leq a < b < \infty$, $\alpha, \beta > 0$ and $g : [0,\infty) \to \mathbf{R}$ be a continuous convex function, ψ a positive differentiable increasing function. Then the following inequalities hold:*

$$g\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\beta+1)}{2\Omega(\alpha,\psi)} \left[{}^{\psi}_{\mathscr{CF}}I^{\beta}_{(\frac{a+b}{2})^-} G(a) \ + \ {}^{\psi}_{\mathscr{CF}}I^{\beta}_{(\frac{a+b}{2})^+} G(b)\right] \leq \frac{g(a)+g(b)}{2} \ , \tag{2.1}$$

where

$$G(t) = g(t) + g(a+b-t), \tag{2.2}$$

and

$$\Omega(\psi,\alpha) = \left(\frac{\psi^{\alpha}(b) - \psi^{\alpha}\left(\frac{a+b}{2}\right)}{\alpha}\right)^{\beta} + \left(\frac{\psi^{\alpha}\left(\frac{a+b}{2}\right) - \psi^{\alpha}(a)}{\alpha}\right)^{\beta}. \tag{2.3}$$

Proof. Let g be a convex function; for any $t \in [[a,b]$, we obtain

$$2\, g\left(\frac{a+b}{2}\right) = 2\, g\left(\frac{a+b-t}{2} + \frac{t}{2}\right) \leq g(t) + g(a+b-t),$$

then

$$2\, g\left(\frac{a+b}{2}\right) \leq G(t). \tag{2.4}$$

Multiplying (2.4) with $\left(\dfrac{\psi^{\alpha}(b) - \psi^{\alpha}(t)}{\alpha}\right)^{\beta-1} \psi'(t)\ \psi^{\alpha-1}(t)$ and integrating over $t \in [\frac{a+b}{2}, b]$, we deduce

$$2\, g\left(\frac{a+b}{2}\right) \int_{\frac{a+b}{2}}^b \left(\frac{\psi^{\alpha}(b) - \psi^{\alpha}(t)}{\alpha}\right)^{\beta-1} \psi'(t)\ \psi^{\alpha-1}(t)\, dt$$

$$\leq \int_{\frac{a+b}{2}}^b \psi'(t) \ \left(\frac{\psi^{\alpha}(b) - \psi^{\alpha}(t)}{\alpha}\right)^{\beta-1} G(t)\ \psi^{\alpha-1}(t)\, dt.$$

This results in

$$g\left(\frac{a+b}{2}\right) \left(\frac{\psi^{\alpha}(b) - \psi^{\alpha}\left(\frac{a+b}{2}\right)}{\alpha}\right)^{\beta} \leq \frac{\Gamma(\beta+1)}{2}\, {}^{\psi}_{\mathscr{CF}}I^{\beta}_{(\frac{a+b}{2})^+} G(b). \tag{2.5}$$

Now, multiplying (2.4) by $\left(\frac{\psi^{\alpha}(t)-\psi^{\alpha}(a)}{\alpha}\right)^{\beta-1}\psi^{'}(t)\ \psi^{\alpha-1}(t)$ and integrating over $t\in\left[a,\frac{a+b}{2}\right]$ gives

$$2\,g\left(\frac{a+b}{2}\right)\int_{a}^{\frac{a+b}{2}}\left(\frac{\psi^{\alpha}(t)-\psi^{\alpha}(a)}{\alpha}\right)^{\beta-1}\psi^{'}(t)\ \psi^{\alpha-1}(t)\ dt$$
$$\leq\int_{a}^{\frac{a+b}{2}}\left(\frac{\psi^{\alpha}(t)-\psi^{\alpha}(a)}{\alpha}\right)^{\beta-1}\psi^{'}(t)\ \psi^{\alpha-1}(t)\ G(t)\ dt\ ,$$

and consequently

$$g\left(\frac{a+b}{2}\right)\left(\frac{\psi^{\alpha}(\frac{a+b}{2})-\psi^{\alpha}(a)}{\alpha}\right)^{\beta}\leq\frac{\Gamma(\beta+1)}{2}\ {}^{\psi}_{\mathscr{CF}}I^{\beta}_{(\frac{a+b}{2})^{-}}G(a)\ . \tag{2.6}$$

The addition of (2.5) and (2.6) results in

$$g\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\beta+1)}{2\Omega(\alpha,\psi)}\left[{}^{\psi}_{\mathscr{CF}}I^{\beta}_{(\frac{a+b}{2})^{-}}G(a)\ +\ {}^{\psi}_{\mathscr{CF}}I^{\beta}_{(\frac{a+b}{2})^{+}}G(b)\right]. \tag{2.7}$$

To demonstrate the second inequality in (2.1), we need to put $t=(1-s)a+sb$ in (2.2) and use the fact that g is convex to get

$$G(t)\ \leq\ g(a)+g(b)\,. \tag{2.8}$$

Use the identical procedure as previously on (2.8), resulting in

$$\frac{\Gamma(\beta+1)}{2\Omega(\alpha,\psi)}\left[{}^{\psi}_{\mathscr{CF}}I^{\beta}_{(\frac{a+b}{2})^{-}}G(a)\ +\ {}^{\psi}_{\mathscr{CF}}I^{\beta}_{(\frac{a+b}{2})^{+}}G(b)\right]\leq\frac{g(a)+g(b)}{2}\ . \tag{2.9}$$

At long last, the necessary result is obtained by integrating the inequality (2.7) with (2.9).

□

Remark 2.2. Theorem 2.1 in [2] is established by setting $\alpha=1$ in Theorem 2.1.

Corollary 2.3. *With the hypothesis of Theorem 2.1, depending on the choice of the function ψ, we obtain the following results:*

1. *Taking $\psi(s)=s$, the following new result is obtained.*

$$g\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\beta+1)}{2\Omega_{1}(\alpha)}\left[{}^{\alpha}\mathscr{I}^{\beta}_{(\frac{a+b}{2})^{-}}G(a)+{}^{\alpha}\mathscr{I}^{\beta}_{(\frac{a+b}{2})^{+}}G(b)\right]\leq\frac{g(a)+g(b)}{2}, \tag{2.10}$$

where

$$\Omega_{1}(\alpha)=\left(\frac{b^{\alpha}-\left(\frac{a+b}{2}\right)^{\alpha}}{\alpha}\right)^{\beta}+\left(\frac{\left(\frac{a+b}{2}\right)^{\alpha}-a^{\alpha}}{\alpha}\right)^{\beta}. \tag{2.11}$$

Setting $\alpha=1$ in (2.10), we obtain the inequalities (3) in [8].

2. *Choose $\psi(s)=\ln s$, we obtain the new result shown below.*

$$g\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\beta+1)}{2\Omega_{2}(\alpha)}\left[{}^{\alpha}\mathscr{H}^{\beta}_{(\frac{a+b}{2})^{-}}G(a)+{}^{\alpha}\mathscr{H}^{\beta}_{(\frac{a+b}{2})^{+}}G(b)\right]\leq\frac{g(a)+g(b)}{2}, \tag{2.12}$$

where

$$\Omega_{2}(\alpha)=\left(\frac{\ln^{\alpha}(b)-\ln^{\alpha}\left(\frac{a+b}{2}\right)}{\alpha}\right)^{\beta}+\left(\frac{\ln^{\alpha}\left(\frac{a+b}{2}\right)-\ln^{\alpha}(a)}{\alpha}\right)^{\beta}. \tag{2.13}$$

Using $\alpha=1$ in (2.12), we obtain Corollary 2.3 [2].

3. Trapezoid Type Inequalities

In this part, we show the special outcomes of some trapezoidal-type inequalities that are found using fractional integral operators that are ψ-conformable. In order to accomplish this, we first establish equality in the subsequent lemma.

Lemma 3.1. *Let $g:[a,b]\to\mathbf{R}$ be a differentiable mapping on (a,b) and α,β,ψ are defined as in Theorem 2.1. Then the following identity holds:*

$$\frac{g(a)+g(b)}{2}-\frac{\Gamma(\beta+1)}{2\Omega(\psi,\alpha)}\left[{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a)+{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right]$$
$$=\frac{b-a}{4\Omega(\psi,\alpha)}\int_0^1\left(\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right)\left[g'\left(\frac{s}{2}a+\frac{2-s}{2}b\right)-g'\left(\frac{2-s}{2}a+\frac{s}{2}b\right)\right]ds, \tag{3.1}$$

where $\Omega(\psi,\alpha)$ is defined as in (2.3) and

$$A_{\psi,\alpha}(s)=\left(\frac{\psi^{\alpha}(b)-\psi^{\alpha}\left(\frac{s}{2}a+\frac{2-s}{2}b\right)}{\alpha}\right)^{\beta}+\left(\frac{\psi^{\alpha}\left(\frac{2-s}{2}a+\frac{s}{2}b\right)-\psi^{\alpha}(a)}{\alpha}\right)^{\beta}. \tag{3.2}$$

Proof. Let

$$J_1=\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\left[\left(\frac{\psi^{\alpha}(b)-\psi^{\alpha}(\frac{a+b}{2})}{\alpha}\right)^{\beta}-\left(\frac{\psi^{\alpha}(b)-\psi^{\alpha}(t)}{\alpha}\right)^{\beta}\right]G'(t)dt, \tag{3.3}$$

by employing integration by parts on (3.3) and utilizing (2.2), we obtain

$$\frac{b-a}{2}J_1=\left.\left[\left(\frac{\psi^{\alpha}(b)-\psi^{\alpha}(\frac{a+b}{2})}{\alpha}\right)^{\beta}-\left(\frac{\psi^{\alpha}(b)-\psi^{\alpha}(t)}{\alpha}\right)^{\beta}\right]G(t)\right|_{\frac{a+b}{2}}^{b}$$
$$-\beta\int_{\frac{a+b}{2}}^{b}\left(\frac{\psi^{\alpha}(b)-\psi^{\alpha}(t)}{\alpha}\right)^{\beta-1}\psi'(t)G(t)d_{\psi}t,$$

therefore

$$\frac{b-a}{2}J_1=\left(\frac{\psi^{\alpha}(b)-\psi^{\alpha}(\frac{a+b}{2})}{\alpha}\right)^{\beta}G(b)-\Gamma(\beta+1)\,{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b). \tag{3.4}$$

Similarly, let

$$J_2=\frac{2}{b-a}\int_{a}^{\frac{a+b}{2}}\left[\left(\frac{\psi^{\alpha}(\frac{a+b}{2})-\psi^{\alpha}(a)}{\alpha}\right)^{\beta}-\left(\frac{\psi^{\alpha}(t)-\psi^{\alpha}(a)}{\alpha}\right)^{\beta}\right]G'(t)dt, \tag{3.5}$$

Through integration by parts (3.5), we arrive at

$$\frac{b-a}{2}J_2=-\left(\frac{\psi^{\alpha}(\frac{a+b}{2})-\psi^{\alpha}(a)}{\alpha}\right)^{\beta}G(a)+\Gamma(\beta+1)\,{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a). \tag{3.6}$$

Given that $G(a)=G(b)=g(a)+g(b)$, we can deduce the following from (3.4) and (3.6):

$$\frac{b-a}{2}(J_1-J_2)=$$
$$\Omega(\psi,\alpha)\left(g(a)+g(b)\right)-\Gamma(\beta+1)\left[{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a)+{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right],$$

thus
$$\frac{g(a)+g(b)}{2}-\frac{\Gamma(\beta+1)}{2\Omega(\psi,\alpha)}\left[{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a)+{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right]=\frac{b-a}{4\Omega(\psi,\alpha)}\left(J_1-J_2\right). \tag{3.7}$$

Conversely, since $G'(t)=g'(t)-g'(a+b-t)$, we derive from (3.3)

$$J_1 = \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\left[\left(\frac{\psi^{\alpha}(b)-\psi^{\alpha}(\frac{a+b}{2})}{\alpha}\right)^{\beta}-\left(\frac{\psi^{\alpha}(b)-\psi^{\alpha}(t)}{\alpha}\right)^{\beta}\right]$$
$$\times\left(g'(t)-g'(a+b-t)\right)dt,$$

By changing the variable $t=\frac{s}{2}a+\frac{2-s}{2}b$, we can achieve the following:

$$J_1 = \int_0^1\left[\left(\frac{\psi^{\alpha}(b)-\psi^{\alpha}(\frac{a+b}{2})}{\alpha}\right)^{\beta}-\left(\frac{\psi^{\alpha}(b)-\psi^{\alpha}(\frac{s}{2}a+\frac{2-s}{2}b)}{\alpha}\right)^{\beta}\right]$$
$$\times\left[g'\left(\frac{s}{2}a+\frac{2-s}{2}b\right)-g'\left(\frac{2-s}{2}a+\frac{s}{2}b\right)\right]ds.$$

Similarly, we get from (3.5)

$$J_2 = \frac{2}{b-a}\int_{a}^{\frac{a+b}{2}}\left[\left(\frac{\psi^{\alpha}(\frac{a+b}{2})-\psi^{\alpha}(a)}{\alpha}\right)^{\beta}-\left(\frac{\psi^{\alpha}(t)-\psi^{\alpha}(a)}{\alpha}\right)^{\beta}\right]$$
$$\times\left(g'(t)-g'(a+b-t)\right)dt,$$

by changing the variable $t=\frac{2-s}{2}a+\frac{s}{2}b$, we get

$$J_2 = \int_0^1\left[\left(\frac{\psi^{\alpha}(\frac{a+b}{2})-\psi^{\alpha}(a)}{\alpha}\right)^{\beta}-\left(\frac{\psi^{\alpha}(\frac{2-s}{2}a+\frac{s}{2}b)-\psi^{\alpha}(a)}{\alpha}\right)^{\beta}\right]$$
$$\times\left[g'\left(\frac{2-s}{2}a+\frac{s}{2}b\right)-g'\left(\frac{s}{2}a+\frac{2-s}{2}b\right)\right]ds.$$

Consequently

$$J_1-J_2=\int_0^1\left(\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right)\left[g'\left(\frac{s}{2}a+\frac{2-s}{2}b\right)-g'\left(\frac{2-s}{2}a+\frac{s}{2}b\right)\right]ds. \tag{3.8}$$

Replacing (3.8) in (3.7), we get the desired equality (3.1). □

Remark 3.2. Specialized cases are available below.

1. Put $\alpha=1$, we get Lemma 4.1 in [2].
2. Put $\psi(t)=t$, we have the following identity for Katugompola fractional integrals [5]

$$\frac{g(a)+g(b)}{2}-\frac{\Gamma(\beta+1)}{2\Omega_1(\alpha)}\left[{}^{\alpha}\mathscr{I}^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a)+{}^{\alpha}\mathscr{I}^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right]=\frac{b-a}{4\Omega_1(\alpha)}$$
$$\times\int_0^1\left(\Omega_1(\alpha)-A_{1,\alpha}(s)\right)\left[g'\left(\frac{s}{2}a+\frac{2-s}{2}b\right)-g'\left(\frac{2-s}{2}a+\frac{s}{2}b\right)\right]ds,$$

where $\Omega_1(\alpha)$ is defined by (2.11) and

$$A_{1,\alpha}(s)=\left(\frac{b^{\alpha}-\left(\frac{s}{2}a+\frac{2-s}{2}b\right)^{\alpha}}{\alpha}\right)^{\beta}+\left(\frac{\left(\frac{2-s}{2}a+\frac{s}{2}b\right)^{\alpha}-a^{\alpha})}{\alpha}\right)^{\beta}. \tag{3.9}$$

3. The following identity for α-Hadamard type-fractional integrals is obtained by setting $\psi(t)=\ln t$.

$$\frac{g(a)+g(b)}{2}-\frac{\Gamma(\beta+1)}{2\Omega_2(\alpha)}\left[{}^{\alpha}\mathscr{H}^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a)+{}^{\alpha}\mathscr{H}^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right]=\frac{b-a}{4\Omega_2(\alpha)}$$

$$\times\int_0^1\left(\Omega_2(\alpha)-A_{2,\alpha}(s)\right)\left[g'\left(\frac{s}{2}a+\frac{2-s}{2}b\right)-g'\left(\frac{2-s}{2}a+\frac{s}{2}b\right)\right]ds,$$

where $\Omega_2(\alpha)$ is defined by (2.13) and

$$A_{2,\alpha}(s)=\left(\frac{\ln^{\alpha}b-\ln^{\alpha}\left(\frac{s}{2}a+\frac{2-s}{2}b\right)}{\alpha}\right)^{\beta}+\left(\frac{\ln^{\alpha}\left(\frac{2-s}{2}a+\frac{s}{2}b\right)-\ln^{\alpha}a)}{\alpha}\right)^{\beta}. \tag{3.10}$$

Theorem 3.3. *Assume that α,β,ψ are defined as in Lemma 3.1. If $|g'|$ is a convex mapping on $[a,b]$, then the trapezoid-type inequality is obtained as*

$$\begin{aligned}&\left|\frac{g(a)+g(b)}{2}-\frac{\Gamma(\beta+1)}{2\Omega(\psi,\alpha)}\left[{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a)+{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right]\right|\\&\le\frac{b-a}{4\Omega(\psi,\alpha)}\left[|g'(a)|+|g'(b)|\right]\int_0^1\left|\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right|ds.\end{aligned} \tag{3.11}$$

Proof. Using the absolute value of identity (3.1) and the convexity of $|g'|$, we deduce

$$\left|\frac{g(a)+g(b)}{2}-\frac{\Gamma(\beta+1)}{2\Omega(\psi,\alpha)}\left[{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a)+{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right]\right|$$

$$\le\frac{b-a}{4\Omega(\psi,\alpha)}\int_0^1\left|\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right|$$

$$\times\left[\left|g'\left(\frac{s}{2}a+\frac{2-s}{2}b\right)\right|+\left|g'\left(\frac{2-s}{2}a+\frac{s}{2}b\right)\right|\right]ds$$

$$\le\frac{b-a}{4\Omega(\psi,\alpha)}\int_0^1\left|\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right|$$

$$\times\left[\frac{s}{2}|g'(a)|+\frac{2-s}{2}|g'(b)|+\frac{2-s}{2}|g'(a)|+\frac{s}{2}|g'(b)|\right]ds$$

$$=\frac{b-a}{4\Omega(\psi,\alpha)}\left[|g'(a)|+|g'(b)|\right]\int_0^1\left|\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right|ds.$$

□

Remark 3.4. By assigning $\alpha=1$ in Theorem 3.3, we derive Theorem 4.4 in [2].

Based on the hypothesis of Theorem 3.3 and the selection of the function ψ, it yields the following Corollary 3.5.

Corollary 3.5. 1. *Taking $\psi(s)=s$, we get*

$$\left|\frac{g(a)+g(b)}{2}-\frac{\Gamma(\beta+1)}{2\Omega_1(\alpha)}\left[{}^{\alpha}\mathscr{I}^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a)+{}^{\alpha}\mathscr{I}^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right]\right| \leq \frac{b-a}{4\Omega_1(\alpha)}\left[|g'(a)|+|g'(b)|\right]\int_0^1\left|\Omega_1(\alpha)-A_{1,\alpha}(s)\right|ds. \tag{3.12}$$

By setting $\alpha=1$ in (3.12), we derive Remark 4.5 [2].

2. *By choosing $\psi(s)=\ln s$, we derive*

$$\left|\frac{g(a)+g(b)}{2}-\frac{\Gamma(\beta+1)}{2\Omega_2(\alpha)}\left[{}^{\alpha}\mathscr{H}^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a)+{}^{\alpha}\mathscr{H}^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right]\right| \leq \frac{b-a}{4\Omega_2(\alpha)}\left[|g'(a)|+|g'(b)|\right]\int_0^1\left|\Omega_2(\alpha)-A_{2,\alpha}(s)\right|ds. \tag{3.13}$$

Setting $\alpha=1$ in (3.13) yields Corollary 4.6 [2].

Theorem 3.6. *Let $p>1$, $\frac{1}{q}+\frac{1}{p}=1$ and assume that α,β,ψ are defined as in Lemma 3.1. If $|g'|^p$ is a convex mapping on $[a,b]$, then the following trapezoid type inequalities hold.*

$$\begin{aligned}
&\left|\frac{g(a)+g(b)}{2}-\frac{\Gamma(\beta+1)}{2\Omega(\psi,\alpha)}\left[{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a)+{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right]\right| \\
&\leq \frac{b-a}{4\Omega(\psi,\alpha)}\left(\int_0^1\left|\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right|^q ds\right)^{\frac{1}{q}} \\
&\times\left[\left(\frac{|g'(a)|^p+3|g'(b)|^p}{4}\right)^{\frac{1}{p}}+\left(\frac{3|g'(a)|^p+|g'(b)|^p}{4}\right)^{\frac{1}{p}}\right] \\
&\leq \frac{b-a}{4\Omega(\psi,\alpha)}\left[|g'(a)|+|g'(b)|\right]\left(4\int_0^1\left|\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right|^q ds\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.14}$$

Proof. According to Lemma 3.1 and utilizing the absolute value, we obtain

$$\begin{aligned}
&\left|\frac{g(a)+g(b)}{2}-\frac{\Gamma(\beta+1)}{2\Omega(\psi,\alpha)}\left[{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a)+{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right]\right| \\
&\leq \frac{b-a}{4\Omega(\psi,\alpha)}\int_0^1\left|\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right|\left|g'\left(\frac{s}{2}a+\frac{2-s}{2}b\right)\right|ds \\
&+\frac{b-a}{4\Omega(\psi,\alpha)}\int_0^1\left|\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right|\left|g'\left(\frac{2-s}{2}a+\frac{s}{2}b\right)\right|ds.
\end{aligned}$$

Hölder's inequality gives

$$\left|\frac{g(a)+g(b)}{2}-\frac{\Gamma(\beta+1)}{2\Omega(\psi,\alpha)}\left[{}^{\psi}_{\mathscr{C}\mathscr{F}}I^{\beta}_{\left(\frac{a+b}{2}\right)^{-}}G(a)+{}^{\psi}_{\mathscr{C}\mathscr{F}}I^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}G(b)\right]\right|$$

$$\leq\frac{b-a}{4\Omega(\psi,\alpha)}\left(\int_0^1\left|\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right|^q ds\right)^{\frac{1}{q}}\left(\int_0^1\left|g'\left(\frac{s}{2}a+\frac{2-s}{2}b\right)\right|^p ds\right)^{\frac{1}{p}}$$

$$+\frac{b-a}{4\Omega(\psi,\alpha)}\left(\int_0^1\left|\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right|^q ds\right)^{\frac{1}{q}}\left(\int_0^1\left|g'\left(\frac{2-s}{2}a+\frac{s}{2}b\right)\right|^p ds\right)^{\frac{1}{p}}.$$

Since $|g'|^p$ is a convex function, we get

$$\left|\frac{g(a)+g(b)}{2}-\frac{\Gamma(\beta+1)}{2\Omega(\psi,\alpha)}\left[{}^{\psi}_{\mathscr{C}\mathscr{F}}I^{\beta}_{b^-}G\left(\frac{a+b}{2}\right)+{}^{\psi}_{\mathscr{C}\mathscr{F}}I^{\beta}_{a^+}G\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\frac{b-a}{4\Omega(\psi,\alpha)}\left(\int_0^1\left|\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right|^q ds\right)^{\frac{1}{q}}\left(\frac{\left|g'(a)\right|^p+3\left|g'(b)\right|^p}{4}\right)^{\frac{1}{p}}$$

$$+\frac{b-a}{4\Omega(\psi,\alpha)}\left(\int_0^1\left|\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right|^q ds\right)^{\frac{1}{q}}\left(\frac{3\left|g'(a)\right|^p+\left|g'(b)\right|^p}{4}\right)^{\frac{1}{p}}.$$

This accomplishes the first inequality in 3.14. For the second inequality, notice that for $A,B>0$, $p>1$, we have

$$A^p+B^p\leq(A+B)^p,\text{ and } 1+3^{\frac{1}{p}}\leq 4.$$

Then

$$\left|\frac{g(a)+g(b)}{2}-\frac{\Gamma(\beta+1)}{2\Omega(\psi,\alpha)}\left[{}^{\psi}_{\mathscr{C}\mathscr{F}}I^{\beta}_{b^-}G\left(\frac{a+b}{2}\right)+{}^{\psi}_{\mathscr{C}\mathscr{F}}I^{\beta}_{a^+}G\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\frac{b-a}{4\Omega(\psi,\alpha)}\left(\int_0^1\left|\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right|^q ds\right)^{\frac{1}{q}}4^{1-\frac{1}{p}}\left(\left|g'(a)\right|+\left|g'(b)\right|\right),$$

which yields to the second inequality in (3.14). □

4. Midpoint Type Inequalities

This section formulates a midpoint-type inequality concerning ψ-conformable fractional integral operators, utilizing the identity provided in the following Lemma.

Lemma 4.1. *Under the hypothesis of Lemma 3.1, the following identity holds*

$$\begin{aligned}&\frac{\Gamma(\beta+1)}{2\Omega(\psi,\alpha)}\left[{}^{\psi}_{\mathscr{C}\mathscr{F}}I^{\beta}_{\left(\frac{a+b}{2}\right)^{-}}G(a)+{}^{\psi}_{\mathscr{C}\mathscr{F}}I^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}G(b)\right]-g\left(\frac{a+b}{2}\right)\\&=\frac{b-a}{4\Omega(\psi,\alpha)}\int_0^1 A_{\psi,\alpha}(s)\left[g'\left(\frac{s}{2}a+\frac{2-s}{2}b\right)-g'\left(\frac{2-s}{2}a+\frac{s}{2}b\right)\right]ds.\end{aligned}\tag{4.1}$$

Proof. Let
$$K_1 = \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\left(\frac{\psi^{\alpha}(b)-\psi^{\alpha}(t)}{\alpha}\right)^{\beta} G'(t)dt, \tag{4.2}$$
By applying integration by parts on (4.2) and using (2.2), we are able to derive
$$\frac{b-a}{2}K_1 = \left(\frac{\psi^{\alpha}(b)-\psi^{\alpha}(t)}{\alpha}\right)^{\beta} G(t)\Bigg|_{\frac{a+b}{2}}^{b} + \beta\int_{\frac{a+b}{2}}^{b}\left(\frac{\psi^{\alpha}(b)-\psi^{\alpha}(t)}{\alpha}\right)^{\beta-1} \psi'(t)G(t)d_{\psi}t,$$
thus
$$\frac{b-a}{2}K_1 = -2\left(\frac{\psi^{\alpha}(b)-\psi^{\alpha}(\frac{a+b}{2})}{\alpha}\right)^{\beta} g\left(\frac{a+b}{2}\right) + \Gamma(\beta+1)\, {}^{\psi}_{\mathscr{C}\mathscr{F}}I^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}G(b). \tag{4.3}$$
Similarly, let
$$K_2 = \frac{2}{b-a}\int_{a}^{\frac{a+b}{2}}\left(\frac{\psi^{\alpha}(t)-\psi^{\alpha}(a)}{\alpha}\right)^{\beta} G'(t)dt. \tag{4.4}$$
Integrating by parts gets
$$\frac{b-a}{2}K_2 = 2\left(\frac{\psi^{\alpha}\left(\frac{a+b}{2}\right)-\psi^{\alpha}(a)}{\alpha}\right)^{\beta} g\left(\frac{a+b}{2}\right) - \Gamma(\beta+1)\, {}^{\psi}_{\mathscr{C}\mathscr{F}}I^{\beta}_{\left(\frac{a+b}{2}\right)^{-}}G(a). \tag{4.5}$$
The sum of (4.3) and (4.5) produces
$$\frac{b-a}{4\Omega(\psi,\alpha)}(K_1-K_2) = \frac{\Gamma(\beta+1)}{2\Omega(\psi,\alpha)}\left[{}^{\psi}_{\mathscr{C}\mathscr{F}}I^{\beta}_{\left(\frac{a+b}{2}\right)^{-}}G(a) + {}^{\psi}_{\mathscr{C}\mathscr{F}}I^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}G(b)\right] - g\left(\frac{a+b}{2}\right). \tag{4.6}$$
Also, we observe from (4.2) and (2.2) that
$$K_1 = \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\left(\frac{\psi^{\alpha}(b)-\psi^{\alpha}(t)}{\alpha}\right)^{\beta}[g'(t)-g'(a+b-t)]\,dt$$
$$= \int_0^1\left(\frac{\psi^{\alpha}(b)-\psi^{\alpha}(\frac{s}{2}a+\frac{2-s}{2}b)}{\alpha}\right)^{\beta}\left[g'\left(\frac{s}{2}a+\frac{1-s}{2}b\right)-g'\left(\frac{2-s}{2}a+\frac{1+s}{2}b\right)\right]ds,$$
as well, by (4.4) and (2.2), we have
$$K_2 = \int_0^1\left(\frac{\psi^{\alpha}(\frac{1-s}{2}a+\frac{1+s}{2}b)-\psi^{\alpha}(\frac{a+b}{2})}{\alpha}\right)^{\beta}\left[g'\left(\frac{2-s}{2}a+\frac{s}{2}b\right)-g'\left(\frac{s}{2}a+\frac{2-s}{2}b\right)\right]ds.$$
Hence
$$K_1-K_2 = \int_0^1 A_{\psi,\alpha}(s)\left[g'\left(\frac{s}{2}a+\frac{1-s}{2}b\right)-g'\left(\frac{2-s}{2}a+\frac{1+s}{2}b\right)\right]ds. \tag{4.7}$$
The desired equality (4.1) is obtained by replacing (4.7) in (4.6).

□

Remark 4.2. Here are several case studies of specific situations.

1. Setting $\alpha = 1$, we get Lemma 3.1 in [2].

2. Put $\psi(t)=t$, we have the following identity for Katugompola fractional operators

$$\frac{\Gamma(\beta+1)}{2\Omega_1(\alpha)}\left[{}^{\alpha}\mathscr{I}^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a)+{}^{\alpha}\mathscr{I}^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right]-g\left(\frac{a+b}{2}\right)$$

$$=\frac{b-a}{4\Omega_1(\alpha)}\int_0^1 A_{1,\alpha}(s)\left[g'\left(\frac{s}{2}a+\frac{2-s}{2}b\right)-g'\left(\frac{2-s}{2}a+\frac{s}{2}b\right)\right]ds.$$

For $\alpha=1$, we deduce remark 3.2 [2].

3. The following identity for α-Hadamard fractional operators is obtained by setting $\psi(t)=\ln t$.

$$\frac{\Gamma(\beta+1)}{2\Omega_2(\alpha)}\left[{}^{\alpha}\mathscr{H}^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a)+{}^{\alpha}\mathscr{H}^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right]-g\left(\frac{a+b}{2}\right)$$

$$=\frac{b-a}{4\Omega_2(\alpha)}\int_0^1 A_{2,\alpha}(s)\left[g'\left(\frac{s}{2}a+\frac{2-s}{2}b\right)-g'\left(\frac{2-s}{2}a+\frac{s}{2}b\right)\right]ds.$$

For $\alpha=1$, we deduce Corollary 3.3 [2].

Theorem 4.3. *Assume that α,β,ψ are defined as in Lemma 3.1. If $|g'|$ is a convex mapping on $[a,b]$, the midpoint-type inequality is obtained as*

$$\left|\frac{\Gamma(\beta+1)}{2\Omega(\psi,\alpha)}\left[{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a)+{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right]-g\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{4\Omega(\psi,\alpha)}\left(\int_0^1 A_{\psi,\alpha}(s)ds\right)\left[|g'(a)|+|g'(b)|\right]. \tag{4.8}$$

Proof. Using the absolute value of identity (4.1) and the convexity of $|g'|$ function, we have

$$\left|\frac{\Gamma(\beta+1)}{2\Omega(\psi,\alpha)}\left[{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a)+{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right]-g\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{b-a}{4\Omega(\psi,\alpha)}\int_0^1 A_{\psi,\alpha}(s)\left[\left|g'\left(\frac{s}{2}a+\frac{2-s}{2}b\right)\right|+\left|g'\left(\frac{2-s}{2}a+\frac{s}{2}b+\right)\right|\right]ds$$

$$\leq\frac{b-a}{4\Omega(\psi,\alpha)}\int_0^1 A_{\psi,\alpha}(s)\left[\frac{s}{2}|g'(a)|+\frac{2-s}{2}|g'(b)|+\frac{2-s}{2}|g'(a)|+\frac{s}{2}|g'(b)|\right]ds$$

$$=\frac{b-a}{4\Omega(\psi,\alpha)}\left(\int_0^1 A_{\psi,\alpha}(s)ds\right)\left[|g'(a)|+|g'(b)|\right].$$

□

Remark 4.4. Setting $\alpha=1$ in Theorem 4.3, we obtain Theorem 3.4 in [2].

Based on the hypothesis of Theorem 4.3 and the selection of the function ψ, it yields the next Corollary 4.5.

Corollary 4.5. 1. *Taking $\psi(s)=s$, we get*

$$\left|\frac{\Gamma(\beta+1)}{2\Omega_1(\alpha)}\left[{}^{\alpha}\mathscr{I}^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a)+{}^{\alpha}\mathscr{I}^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right]-g\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{b-a}{4\Omega_1(\alpha)}\left(\int_0^1 A_{1,\alpha}(s)ds\right)\left[|g'(a)|+|g'(b)|\right].$$

By setting $\alpha=1$, we derive Remark 3.5 [2].

2. *Choose $\psi(s) = \ln s$. So, we get a novel result with α-Hadamard operators.*

$$\left|\frac{\Gamma(\beta+1)}{2\Omega_2(\alpha)}\left[{}^{\alpha}\mathscr{H}^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a) + {}^{\alpha}\mathscr{H}^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right] - g\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4\Omega_2(\alpha)}\left(\int_0^1 A_{2,\alpha}(s)ds\right)\left[|g'(a)|+|g'(b)|\right]. \tag{4.9}$$

Setting $\alpha = 1$ in (4.9) yields Corollary 3.6 [2].

Theorem 4.6. *Let $p > 1$ and assume that α, β, ψ are defined as in Lemma 3.1. If $|g'|^p$ is a convex mapping on $[a,b]$, then the following midpoint-type inequality is obtained.*

$$\left|\frac{\Gamma(\beta+1)}{2\Omega(\psi,\alpha)}\left[{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a) + {}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right] - g\left(\frac{a+b}{2}\right)\right|$$

$$\le \frac{b-a}{4\Omega(\psi,\alpha)}\left(\int_0^1 A_{\psi,\alpha}(s)ds\right)^{\frac{1}{q}}\left[\left(\frac{|g'(a)|^p+3|g'(b)|^p}{4}\right)^{\frac{1}{p}} + \left(\frac{3|g'(a)|^p+|g'(b)|^p}{4}\right)^{\frac{1}{p}}\right] \tag{4.10}$$

$$\le \frac{b-a}{4\Omega(\psi,\alpha)}\left(4\int_0^1 A_{\psi,\alpha}(s)ds\right)^{\frac{1}{q}}\left[|g'(a)|+|g'(b)|\right],$$

where $\frac{1}{q}+\frac{1}{p} = 1$.

Proof. By Lemma 4.1, using the absolute value, Hölder's inequality and convexity of $|g'|^p$, we deduce

$$\left|\frac{\Gamma(\beta+1)}{2\Omega(\psi,\alpha)}\left[{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a) + {}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right] - g\left(\frac{a+b}{2}\right)\right|$$

$$\le \frac{b-a}{4\Omega(\psi,\alpha)}\left(\int_0^1 A_{\psi,\alpha}(s)ds\right)^{\frac{1}{q}}\left(\int_0^1\left|g'\left(\frac{s}{2}a+\frac{2-s}{2}b\right)\right|^p ds\right)^{\frac{1}{p}}$$

$$+\frac{b-a}{4\Omega(\psi,\alpha)}\left(\int_0^1 A_{\psi,\alpha}(s)ds\right)^{\frac{1}{q}}\left(\int_0^1\left|g'\left(\frac{2-s}{2}a+\frac{s}{2}b\right)\right|^p ds\right)^{\frac{1}{p}}$$

$$\le \frac{b-a}{4\Omega(\psi,\alpha)}\left(\int_0^1 A_{\psi,\alpha}(s)ds\right)^{\frac{1}{q}}\left(\frac{\left|g'(a)\right|^p+3\left|g'(b)\right|^p}{4}\right)^{\frac{1}{p}}$$

$$+\frac{b-a}{4\Omega(\psi,\alpha)}\left(\int_0^1 A_{\psi,\alpha}(s)ds\right)^{\frac{1}{q}}\left(\frac{3\left|g'(a)\right|^p+\left|g'(b)\right|^p}{4}\right)^{\frac{1}{p}}.$$

For the second inequality, use once again $A^p + B^p \le (A+B)^p$ and $1+3^{\frac{1}{p}} \le 4$ for $p > 1$, thus

$$\left|\frac{\Gamma(\beta+1)}{2\Omega(\psi,\alpha)}\left[{}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^-}G(a) + {}^{\psi}_{\mathscr{CF}}I^{\beta}_{\left(\frac{a+b}{2}\right)^+}G(b)\right] - g\left(\frac{a+b}{2}\right)\right|$$

$$\le \frac{b-a}{4\Omega(\psi,\alpha)}\left(\int_0^1 A_{\psi,\alpha}(s)ds\right)^{\frac{1}{q}} 4^{1-\frac{1}{p}} \times \left(\left|g'(a)\right|+\left|g'(b)\right|\right),$$

which yields to the second inequality in (4.10). □

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References

[1] N. Azzouz, B. Benaissa, H. Budak, *On generalized ψ-conformable calculus: properties and inequalities*, Filomat, 38(25), 2024, 8755–8772. DOI: 10.2298/FIL2425755A

[2] H. Budak, *New Hermite–Hadamard-type inequalities for convex mappings utilizing generalized fractional integrals*, Filomat, 33(8), 2019, 2329–2344. DOI: 10.2298/FIL1908329B

[3] M. Jleli and B. Samet, *On Hermite–Hadamard-type inequalities via fractional integrals of a function with respect to another function*, Journal Nonlinear Science and Applications, 9(3), 2016, 1252–1260. DOI: 10.22436/jnsa.009.03.50

[4] H. Kara, S. Erden and H. Budak, *Hermite–Hadamard, trapezoid and midpoint type inequalities involving generalized fractional integrals for convex functions*, Sahand Communications in Mathematical Analysis, 20(2), 2023, 85–107. DOI: 10.22130/scma.2022.552775.1102

[5] U. N. Katugompola, *New approach generalized fractional integral*, Applied Mathematics and Computation, 218(3), 2011, 860–865. DOI: 10.1016/j.amc.2011.03.062

[6] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, *A new definition of fractional derivative*, Journal of Computational Applied Mathematics, 264 (2014), 65–70. DOI: 10.1016/j.cam.2014.01.002

[7] H. Li, B. Meftah, W. Saleh, H. Xu, A. Kiliçman and A. Lakhdari, *Further Hermite–Hadamard-type inequalities for fractional integrals with exponential kernels*, Fractal and Fractional, 8(6), 2024, 345. DOI: 10.3390/fractalfract8060345

[8] M. Z. Sarikaya and H. Yildirim, *On Hermite–Hadamard-type inequalities for Riemann-Liouville fractional integrals*, Miskolc Mathematical Notes, 17(2), 2016, 1049–1059. DOI: 10.18514/MMN.2017.1197

[9] E. Set, J. Choi, and A. Gözpinar, *Hermite–Hadamard-type inequalities involving nonlocal conformable fractional integrals*, Malaysian Journal of Mathematical Sciences, 15(1), 2021, 33–43.