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Research Article

Kantorovich-type Integral Inequalities and Their Fractional Extensions

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Abstract

We establish sharp Kantorovich-type inequalities for Lebesgue integrable functions bounded between two positive constants. By constructing extremal piecewise constant functions and applying classical techniques, we derive precise upper bounds for the ratio of L^2 and L^1 norms and extend the results to weighted settings. Furthermore, we generalize the inequalities to the framework of the Riemann-Liouville fractional integral of order $\alpha \in (0,1]$, capturing nonlocal behaviour and memory effects. These results refine known inequalities and provide sharper analytical tools for applications in classical and fractional analysis.

Keywords: Kantorovich inequality, weighted inequalities, Riemann-Liouville fractional integral

2020 MSC: 26D15, 26A33, 47G20, 39B62.

1. Introduction

Inequalities play a central role in mathematical analysis, with numerous applications in functional analysis, operator theory, and approximation theory. Among the classical contributions, the monumental works of Hardy et. al [1] as well as Mitrinovic et. al [2] have laid the foundation for the systematic study of inequalities involving convexity, means, and integral operators. These texts serve as cornerstones for a wide range of results concerning classical inequalities such as those of Jensen, Hölder, Minkowski, and Cauchy–Schwarz.

A significant milestone in this theory was established by Kantorovich [3], who derived a celebrated inequality providing a sharp relationship between the arithmetic and geometric means under boundedness conditions. The Kantorovich inequality has since been recognized as a powerful tool in operator theory, numerical analysis, and optimization. Its importance was further emphasized and generalized by Greub and Rheinboldt [4], who extended Kantorovich's result to a more general framework of bilinear forms and operator inequalities. Their approach highlighted the structural significance of such inequalities in functional analysis and matrix theory.

In more recent developments, Chansangiam [5] investigated integral inequalities of Kantorovich and Fiedler types for Hadamard products of positive operators, obtaining refined operator inequalities that demonstrate the close interplay between matrix analysis and functional inequalities. Similarly, Zhao and Cheung [6] proposed several improvements of Kantorovich-type inequalities by introducing sharp constants and identifying conditions for equality, thereby contributing to the refinement and precision of such results. Sababheh et al. [7] provided new insights into the

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connection between the Kantorovich and Ando inequalities, offering a unified framework that bridges operator means and functional inequalities. Moreover, Roy and Sain [8] extended the Kantorovich inequality to the setting of Hilbert spaces, illustrating its robustness in abstract inner product spaces and emphasizing its fundamental role in modern analysis.

These studies collectively demonstrate that Kantorovich-type inequalities remain an active area of research, both for their intrinsic theoretical value and for their applications in operator theory, fractional calculus, and integral inequalities. A classical result states that for a Riemann integrable function $f : [\alpha,\beta] \to \mathbb{R}$ satisfying $a \le f(x) \le b$ for all $x \in [\alpha,\beta]$, the inequality

$$\int_\alpha^\beta f^2(x)\,dx \le \frac{(a+b)^2}{4ab}\left(\int_\alpha^\beta f(x)\,dx\right)^2$$

holds (see, e.g., [2]). This inequality, often viewed as an additive form of the Grüss inequality, highlights the connection between integral means and deviation bounds.

The purpose of this paper is to extend, refine, and generalize the classical Kantorovich inequality in several directions. In Section 2, we revisit the classical inequality and derive new sharp integral inequalities for bounded and positive functions. In particular, we establish that for $f : [0,1] \to [m,M]$ with $0 < m \le f(x) \le M$, the following refined inequality holds:

$$\frac{\int_0^1 f^2(x)\,dx}{\left(\int_0^1 f(x)\,dx\right)^2} \le \frac{(M+m)^2}{4Mm},$$

where the constant is shown to be best possible through extremal constructions. Furthermore, we extend this result to a weighted setting by considering a general weight function $\lambda(x)$, which allows the inequality to account for the distributional influence of the weight:

$$\left(\int_a^b \lambda(x) f^2(x)\,dx\right)\left(\int_a^b \frac{\lambda(x)}{f^2(x)}\,dx\right) \le \frac{(\lambda_{\min}+\lambda_{\max})^2}{4\lambda_{\min}\lambda_{\max}}\left(\int_a^b \lambda(x)\,dx\right)^2.$$

In Section 3, we generalize these results to the fractional setting via the Riemann–Liouville fractional integral of order $\alpha \in (0,1]$, defined by

$$I_{a+}^{\alpha}[f](x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt.$$

We establish corresponding Kantorovich-type inequalities in this fractional framework. This extension captures nonlocal memory effects and reduces to the classical case when $\alpha = 1$ and $\lambda(x) = 1$, thereby bridging the classical and fractional theories of integral inequalities.

Finally, in Section 4, we provide concrete numerical illustrations that confirm the sharpness of the established bounds and demonstrate the effectiveness of our results in both classical and fractional cases. These examples support the theoretical findings and highlight the accuracy of the proposed inequalities.

2. Kantorovich-type inequalities for bounded positive functions

Theorem 2.1. *Let $f : [0,1] \to [m,M]$ be a Lebesgue integrable function such that $0 < m \le f(x) \le M < \infty$ for all $x \in [0,1]$. Then the following inequality holds:*

$$\frac{\int_0^1 f^2(x)\,dx}{\left(\int_0^1 f(x)\,dx\right)^2} \le \frac{(M+m)^2}{4Mm},$$

and the constant $\dfrac{(M+m)^2}{4Mm}$ *is the best possible.*

Proof. Define a piecewise constant function f_α for $\alpha \in [0,1]$ by

$$f_\alpha(x) = \begin{cases} M, & \text{if } x \in [0,\alpha], \\ m, & \text{if } x \in (\alpha,1]. \end{cases}$$

Then, we compute:

$$\int_0^1 f_\alpha(x)\,dx = \alpha M + (1-\alpha)m,$$

$$\int_0^1 f_\alpha^2(x)\,dx = \alpha M^2 + (1-\alpha)m^2.$$

Hence,

$$\phi(\alpha) := \frac{\int_0^1 f_\alpha^2(x)\,dx}{\left(\int_0^1 f_\alpha(x)\,dx\right)^2} = \frac{\alpha M^2 + (1-\alpha)m^2}{(\alpha M + (1-\alpha)m)^2}.$$

To find the maximum of $\phi(\alpha)$ over $\alpha \in [0,1]$, consider the value at

$$\alpha = \frac{m}{m+M}.$$

Then,

$$\int_0^1 f_\alpha(x)\,dx = \frac{m}{m+M}M + \frac{M}{m+M}m = \frac{2mM}{m+M},$$

$$\left(\int_0^1 f_\alpha(x)\,dx\right)^2 = \frac{4m^2M^2}{(m+M)^2},$$

$$\int_0^1 f_\alpha^2(x)\,dx = \frac{m}{m+M}M^2 + \frac{M}{m+M}m^2 = \frac{M^2m + m^2M}{m+M} = \frac{mM(M+m)}{m+M} = mM.$$

Thus,

$$\phi(\alpha) = \frac{mM}{\frac{4m^2M^2}{(m+M)^2}} = \frac{(m+M)^2}{4mM}.$$

This shows that the maximum value of the expression is $\frac{(M+m)^2}{4Mm}$, and hence

$$\frac{\int_0^1 f^2(x)\,dx}{\left(\int_0^1 f(x)\,dx\right)^2} \le \frac{(M+m)^2}{4Mm}.$$

The equality holds for piecewise constant functions of the above form. To show that the constant $\frac{(M+m)^2}{4Mm}$ is the best possible, assume that there exists a smaller constant $C < \frac{(M+m)^2}{4Mm}$ such that

$$\frac{\int_0^1 f^2(x)\,dx}{\left(\int_0^1 f(x)\,dx\right)^2} \le C$$

holds for all admissible functions f. However, for the piecewise constant functions f_α defined above with $\alpha = \frac{m}{m+M}$, we have equality at

$$\frac{\int_0^1 f_\alpha^2(x)\,dx}{\left(\int_0^1 f_\alpha(x)\,dx\right)^2} = \frac{(M+m)^2}{4Mm} > C,$$

which contradicts the assumed inequality. Hence, no smaller constant than $\frac{(M+m)^2}{4Mm}$ can satisfy the inequality for all such functions. □

Theorem 2.2. *Let $f : [a,b] \to [m,M]$ be a Lebesgue integrable function such that $0 < m \leq f(x) \leq M < \infty$ for all $x \in [a,b]$. Let $\lambda : [a,b] \to (0,\infty)$ be an integrable weight function, and define*

$$\lambda_{\min} := \inf_{x\in[a,b]} \lambda(x), \quad \lambda_{\max} := \sup_{x\in[a,b]} \lambda(x).$$

Then the following inequality holds:

$$\left(\int_a^b \lambda(x) f^2(x)\,dx\right) \cdot \left(\int_a^b \frac{\lambda(x)}{f^2(x)}\,dx\right) \leq \frac{(\lambda_{\min}+\lambda_{\max})^2}{4\lambda_{\min}\lambda_{\max}} \left(\int_a^b \lambda(x)\,dx\right)^2.$$

Proof. Since $f(x) \in [m,M]$, it follows that $\frac{1}{f^2(x)} \in \left[\frac{1}{M^2}, \frac{1}{m^2}\right]$. Let us define:

$$I_1 := \int_a^b \lambda(x) f^2(x)\,dx, \quad I_2 := \int_a^b \frac{\lambda(x)}{f^2(x)}\,dx, \quad L := \int_a^b \lambda(x)\,dx.$$

Now define the function $h(x) := f^2(x)$, which takes values in $[m^2, M^2]$. Then:

$$I_1 = \int_a^b \lambda(x) h(x)\,dx, \quad I_2 = \int_a^b \lambda(x) \cdot \frac{1}{h(x)}\,dx.$$

We now apply the Kantorovich inequality for positive functions $h(x) \in [m^2, M^2]$ with weights $\lambda(x)$. According to a known result (see [2]), we have:

$$\left(\int_a^b \lambda(x) h(x)\,dx\right) \cdot \left(\int_a^b \lambda(x) \cdot \frac{1}{h(x)}\,dx\right) \leq \frac{(\lambda_{\min}+\lambda_{\max})^2}{4\lambda_{\min}\lambda_{\max}} \left(\int_a^b \lambda(x)\,dx\right)^2.$$

Substituting back $h(x) = f^2(x)$, we obtain:

$$\left(\int_a^b \lambda(x) f^2(x)\,dx\right) \cdot \left(\int_a^b \frac{\lambda(x)}{f^2(x)}\,dx\right) \leq \frac{(\lambda_{\min}+\lambda_{\max})^2}{4\lambda_{\min}\lambda_{\max}} \left(\int_a^b \lambda(x)\,dx\right)^2.$$

This completes the proof. □

3. A Kantorovich-type inequality for the Riemann–Liouville integral

Let $\alpha > 0$. The *left-sided Riemann–Liouville fractional integral* of order α of a function φ defined on $[a,b]$ is given by

$$I_{a+}^{\alpha}[\varphi](x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \varphi(t)\,dt, \quad x \in (a,b],$$

provided that the integral exists. Here, $\Gamma(\cdot)$ denotes the classical Gamma function. The following inequality generalizes the classical Kantorovich inequality to the Riemann-Liouville fractional integral setting, while preserving the best possible constant.

Theorem 3.1. *Let $f : [a,b] \to [m,M]$ be a Lebesgue integrable function such that $0 < m \leq f(x) \leq M < \infty$ for all $x \in [a,b]$. Let $\lambda : [a,b] \to (0,\infty)$ be a Lebesgue integrable weight function. Fix $\alpha \in (0,1]$ and define:*

$$\lambda_{\min} := \inf_{x\in[a,b]} \lambda(x), \quad \lambda_{\max} := \sup_{x\in[a,b]} \lambda(x).$$

Then the following inequality holds:

$$I_{a+}^{\alpha}[\lambda(x) f^2(x)](b) \cdot I_{a+}^{\alpha}\left[\frac{\lambda(x)}{f^2(x)}\right](b) \leq \frac{(\lambda_{\min}+\lambda_{\max})^2}{4\lambda_{\min}\lambda_{\max}} \left(I_{a+}^{\alpha}[\lambda(x)](b)\right)^2.$$

Proof. Let us define the weight function

$$w(x) := (b-x)^{\alpha-1}\lambda(x), \quad x \in [a,b].$$

Observe that since $\lambda(x) > 0$ and $(b-x)^{\alpha-1} > 0$ for all $x \in [a,b)$, we have $w(x) > 0$ on $[a,b)$. Also, w is integrable because λ is integrable and $(b-x)^{\alpha-1}$ is locally integrable on $[a,b]$ for $\alpha \in (0,1]$.

Define the expressions:

$$I := \int_a^b w(x) f^2(x)\,dx = \Gamma(\alpha) I_{a+}^{\alpha}[\lambda(x) f^2(x)](b),$$

$$J := \int_a^b \frac{w(x)}{f^2(x)}\,dx = \Gamma(\alpha) I_{a+}^{\alpha}\left[\frac{\lambda(x)}{f^2(x)}\right](b),$$

$$L := \int_a^b w(x)\,dx = \Gamma(\alpha) I_{a+}^{\alpha}[\lambda(x)](b).$$

Since $f(x) \in [m,M]$, it follows that $f^2(x) \in [m^2, M^2]$ and hence $1/f^2(x) \in [1/M^2, 1/m^2]$.

We now apply the classical weighted Kantorovich inequality for positive functions and positive weights (see [2]), which yields:

$$I \cdot J \leq \frac{(\lambda_{\min} + \lambda_{\max})^2}{4\lambda_{\min}\lambda_{\max}} L^2.$$

Substituting the definitions of I, J, and L gives:

$$\Gamma(\alpha)^2 \cdot I_{a+}^{\alpha}[\lambda(x) f^2(x)](b) \cdot I_{a+}^{\alpha}\left[\frac{\lambda(x)}{f^2(x)}\right](b) \leq \frac{(\lambda_{\min} + \lambda_{\max})^2}{4\lambda_{\min}\lambda_{\max}} \cdot \Gamma(\alpha)^2 \cdot \left(I_{a+}^{\alpha}[\lambda(x)](b)\right)^2.$$

Finally, cancelling out $\Gamma(\alpha)^2$ from both sides yields the desired inequality:

$$I_{a+}^{\alpha}[\lambda(x) f^2(x)](b) \cdot I_{a+}^{\alpha}\left[\frac{\lambda(x)}{f^2(x)}\right](b) \leq \frac{(\lambda_{\min} + \lambda_{\max})^2}{4\lambda_{\min}\lambda_{\max}} \left(I_{a+}^{\alpha}[\lambda(x)](b)\right)^2.$$

This completes the proof. □

Corollary 3.2. *Let $f : [a,b] \to [m,M] \subset (0,\infty)$ be a Lebesgue integrable function and let $\alpha \in (0,1]$. Then the following inequality holds:*

$$\left(I_{a+}^{\alpha}[f^2](b)\right) \cdot \left(I_{a+}^{\alpha}\left[\frac{1}{f^2}\right](b)\right) \leq \frac{(M+m)^2}{4Mm} \left(I_{a+}^{\alpha}[1](b)\right)^2,$$

where the left-sided Riemann–Liouville fractional integral of order $\alpha \in (0,1]$ is defined by

$$I_{a+}^{\alpha}[\varphi](b) = \frac{1}{\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1} \varphi(x)\,dx.$$

Proof. Set the weight function $\lambda(x) = 1$, which is clearly positive and satisfies $m_\lambda = M_\lambda = 1$. Then, applying the main Kantorovich-type inequality from Theorem 3.1, we obtain

$$\left(I_{a+}^{\alpha}[f^2](b)\right) \cdot \left(I_{a+}^{\alpha}\left[\frac{1}{f^2}\right](b)\right) \leq \frac{(M+m)^2}{4Mm} \left(I_{a+}^{\alpha}[1](b)\right)^2.$$

This completes the proof. □

Corollary 3.3 (Classical Kantorovich Inequality)**.** *Let $f : [a,b] \to [m,M] \subset (0,\infty)$ be a Lebesgue integrable function. Then the following inequality holds:*

$$\left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b \frac{1}{f^2(x)}\,dx\right) \leq \frac{(M+m)^2}{4Mm}(b-a)^2.$$

Proof. Since $f(x) \in [m,M]$ for all $x \in [a,b]$, it follows that $m^2 \le f^2(x) \le M^2$ and $\frac{1}{M^2} \le \frac{1}{f^2(x)} \le \frac{1}{m^2}$.

By the classical Kantorovich inequality applied to the pair $f^2(x)$ and $1/f^2(x)$, both positive and bounded, we obtain:

$$\left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b \frac{1}{f^2(x)}\,dx\right) \le \frac{(M+m)^2}{4Mm}(b-a)^2.$$

□

4. Numerical example

Consider the function

$$f(x) = 2 + \frac{1}{2}\sin x,$$

defined on the interval $[0,\pi]$. Since $\sin x \in [-1,1]$, we have $1.5 \le f(x) \le 2.5$, and therefore $m = 1.5$, $M = 2.5$.

Let the weight function be $\lambda(x) = 1 + \frac{1}{2}\cos x$, which satisfies

$$0.5 \le \lambda(x) \le 1.5,$$

thus $\lambda_{\min} = 0.5$ and $\lambda_{\max} = 1.5$.

The integral of the weight function over $[0,\pi]$ is

$$\int_0^\pi \lambda(x)\,dx = \int_0^\pi \left(1 + \frac{1}{2}\cos x\right)dx = \pi + \frac{1}{2}\int_0^\pi \cos x\,dx = \pi + \frac{1}{2}(\sin\pi - \sin 0) = \pi.$$

Applying the Kantorovich-type inequality from Theorem 2.2 yields

$$\left(\int_0^\pi \lambda(x) f^2(x)\,dx\right)\cdot\left(\int_0^\pi \frac{\lambda(x)}{f^2(x)}\,dx\right) \le \frac{(\lambda_{\min}+\lambda_{\max})^2}{4\lambda_{\min}\lambda_{\max}}\left(\int_0^\pi \lambda(x)\,dx\right)^2.$$

Calculating the coefficient,

$$\frac{(0.5+1.5)^2}{4\times 0.5\times 1.5} = \frac{(2.0)^2}{3.0} = \frac{4}{3} \approx 1.3333,$$

and thus

$$\left(\int_0^\pi \lambda(x) f^2(x)\,dx\right)\cdot\left(\int_0^\pi \frac{\lambda(x)}{f^2(x)}\,dx\right) \le 1.3333\times\pi^2 \approx 1.3333\times 9.8696 \approx 13.16.$$

This explicit bound holds exactly under the assumptions of positivity and boundedness of f and λ, illustrating a precise and practical use of the Kantorovich-type inequality.

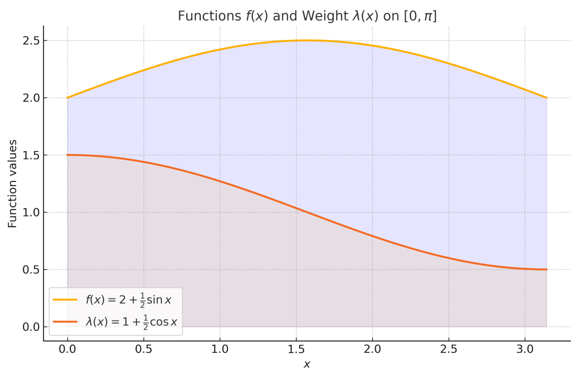

Figure 1: Graphs of the functions $f(x) = 2 + \frac{1}{2}\sin x$ (blue) and $\lambda(x) = 1 + \frac{1}{2}\cos x$ (orange) on the interval $[0,\pi]$. Both functions are positive and bounded, satisfying the assumptions of the Kantorovich-type inequality.

5. Conclusion

In this paper, we have established sharp Kantorovich-type integral inequalities for positive bounded functions, incorporating both classical and fractional integral frameworks. In particular, we extended the inequality to the setting of the Riemann–Liouville fractional integral, providing a broader generalization with memory effects controlled by the order $\alpha \in (0, 1]$. The obtained results refine the classical Kantorovich inequality and reduce to known forms when the fractional parameter and weight function are specialized. The sharpness of the constants was verified via explicit examples. These findings contribute to the theory of fractional inequalities and can be applied in various areas of analysis, including fractional differential equations and integral operator theory.

Use of AI Tools

AI tools were not employed in generating, analyzing, or interpreting the results.

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