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Research Article

The Integral Inequalities Related to HT-convex Functions

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Abstract

In the work, using Hölder's integral inequality and arithmetic-geometric-harmonic inequality, the author establishes some new Hermite–Hadamard type integral inequalities related to differentiable HT-convex functions and some new integral inequalities related to the product of MT-convex and HT-convex functions.

Keywords: Integral inequality, Hermite–Hadamard inequality, HT-convex function, MT-convex function

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1. Introduction

The convex function was introduced by Jensen in 1905 which is one of fundamental concepts in fields such as analysis, geometry, and functional analysis. Their properties, such as Jensen's inequality and sub-differentiability, provide key tools for studying functional inequalities, variational problems, etc. The following definition is the concept of convex function:

Definition 1.1 ([1], [2])**.** A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

The following Hermite-Hadamard inequality is a classical inequality for convex functions and reveals the relationship between the integral mean value of a convex function on an interval and the function values at the endpoints and midpoints.

Theorem 1.2 ([3])**.** *If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I and $a, b \in I$ with $a < b$, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}.$$

In 2012, Tunç and Yildirim introduced a type of generalized convex function in [4].

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Definition 1.3 ([4]). Let $I \subseteq \mathbb{R}$ be an interval. A nonnegative function $f : I \to \mathbb{R}_0 = [0,\infty)$ is said to be MT-convex if the inequality

$$f(tx+(1-t)y) \le \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}f(y)$$

holds for all $x, y \in I$ and $t \in (0,1)$.

In [5], the concept of HT-convex functions below was innovated.

Definition 1.4 ([5]). Let $I \subseteq \mathbb{R}\backslash\{0\}$ be an interval. A function $f : I \to \mathbb{R}_0$ is called HT-convex function if the inequality

$$f\left(\frac{xy}{ty+(1-t)x}\right) \le \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}f(y)$$

holds for all $x, y \in I$ and $t \in (0,1)$.

With the extension of convex functions, Hermite-Hadamard type inequalities for MT-convex functions and HT-convex functions have also been studied and established.

Theorem 1.5 ([4, Theorem 2]). *Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}_0$ is* MT*-convex on I, $a, b \in I$ with $a < b$ and $f \in L_1([a,b])$. Then*

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(x)\,dx \quad \text{and} \quad \frac{2}{b-a}\int_a^b \tau(x)f(x)\,dx \le \frac{f(a)+f(b)}{2},$$

where

$$\tau(x) = \frac{\sqrt{(b-x)(x-a)}}{b-a}, \quad x \in [a,b].$$

Theorem 1.6 ([5, Theorem 4.1]). *Suppose that $I \subseteq \mathbb{R}\backslash\{0\}$ is an real interval. A function $f : I \to \mathbb{R}_0$ is* HT*-convex on I, $a, b \in I$ with $a < b$ and $f \in L_1([a,b])$, then*

$$f\Big(\frac{2ab}{a+b}\Big) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx \le \frac{\pi}{4}\big[f(a)+f(b)\big].$$

In recent decades, a lot of inequalities of Hermite–Hadamard type for various generalized convex functions have been established, for example, MT-h-convex functions, coordinated MT-(s_1,s_2)-convex functions, GT-convex functions and geometrically P-convex functions, etc. All details can be found in [6–8].

Inspired by the above research work, we will establish some new Hermite–Hadamard type integral inequalities related to differentiable HT-convex functions by using Hölder's integral inequality. Moreover, we will also derive some new integral inequalities for the product of MT-convex and HT-convex functions by using arithmetic-geometric-harmonic inequality.

2. Several Lemmas

From the Lemma 1 in [9], we get

Lemma 2.1 ([9]). *Let $f : I \subseteq \mathbb{R}_+ := (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L_1([a,b])$, then*

$$\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx = \frac{b-a}{4ab}\int_0^1 \Big[tH_t^2(b,H(a,b))f'\left(H_t(b,H(a,b))\right) - tH_t^2(a,H(a,b))f'\left(H_t(a,H(a,b))\right)\Big]\,dt,$$

where the harmonic mean $H(x,y)$ and the weight harmonic mean $H_t(x,y)$ are defined by

$$H(x,y) = \frac{2xy}{x+y} \quad \text{and} \quad H_t(x,y) = \left(\frac{t}{x} + \frac{1-t}{y}\right)^{-1} = \frac{xy}{ty+(1-t)x}$$

for $x, y \in \mathbb{R}_+$ and $t \in [0,1]$.

Lemma 2.2 ([9]). *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ with $a < b$. If $f' \in L_1([a,b])$, then*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx - f\left(\frac{2ab}{a+b}\right) = \frac{b-a}{4ab}\int_0^1 (1-t)\Big[H_t^2\big(b,H(a,b)\big)f'\left(H_t\big(b,H(a,b)\big)\right) - H_t^2\big(a,H(a,b)\big)f'\left(H_t\big(a,H(a,b)\big)\right)\Big]\,dt,$$

where $H(x,y)$, $H_t(x,y)$ are the harmonic mean and the weight harmonic mean, respectively.

Lemma 2.3. *Let $u,v,x,y \in \mathbb{R}_+$ and $p \geq 1$ with $u \neq v$, then*

$$S_p(u,v) := \int_0^1 tH_t^{2p}(u,v)\,dt = \begin{cases} \dfrac{u^2v\left[(\ln v - \ln u)v + u - v\right]}{(v-u)^2}, & p = 1, \\[2ex] \dfrac{u^2v\left[v^{2p-1} - (2p-1)u^{2(p-1)}v + 2(p-1)u^{2p-1}\right]}{2(p-1)(2p-1)(v-u)^2}, & p > 1, \end{cases}$$

and

$$T\big(u,v;x,y\big) := \int_0^1 \frac{\sqrt{t(1-t)^{-1}}}{2}H_t^2(u,v)A_t\big(x,y\big)\,dt = \frac{\pi u\sqrt{uv}}{4(v-u)^2}\left\{\left[u^2 + \left(2\sqrt{uv} - 3u\right)v\right]x + v\left(\sqrt{v} - \sqrt{u}\right)^2 y\right\},$$

where $A_t(x,y) := tx + (1-t)y$ is the weight arithmetic mean and $H_t(x,y)$ is the weight harmonic mean for $x,y \in \mathbb{R}_+$ and $t \in [0,1]$.

3. Hermite–Hadamard type integral inequalities related to differentiable HT-convex functions

Theorem 3.1. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ with $a < b$, and $f' \in L_1([a,b])$. If $|f'|^q$ is an* HT*-convex function on $[a,b]$ for $q \geq 1$, then*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx\right| \\ &\leq \frac{b-a}{4ab}\left[S_1(a,H(a,b))\right]^{1-\frac{1}{q}}\left[T\big(a,H(a,b);|f'(a)|^q,|f'(H(a,b))|^q\big)\right]^{\frac{1}{q}} \\ &\quad + \frac{b-a}{4ab}\left[S_1(b,H(a,b))\right]^{1-\frac{1}{q}}\left[T\big(b,H(a,b);|f'(b)|^q,|f'(H(a,b))|^q\big)\right]^{\frac{1}{q}}, \end{aligned} \tag{3.1}$$

where $H(x,y)$ is the harmonic mean, $S_1(u,v)$ and $T\big(u,v;x,y\big)$ are defined as in Lemma 2.3.

Proof. From Lemma 2.1 and Hölder's integral inequality, we acquire

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx\right| \\ &\leq \frac{b-a}{4ab}\int_0^1 \Big[tH_t^2\big(b,H(a,b)\big)\left|f'\left(H_t\big(b,H(a,b)\big)\right)\right| + tH_t^2\big(a,H(a,b)\big)\left|f'\left(H_t\big(a,H(a,b)\big)\right)\right|\Big]\,dt \\ &\leq \frac{b-a}{4ab}\left[\int_0^1 tH_t^2\big(a,H(a,b)\big)\,dt\right]^{1-\frac{1}{q}}\left[\int_0^1 tH_t^2\big(a,H(a,b)\big)\left|f'\left(H_t\big(a,H(a,b)\big)\right)\right|^q\,dt\right]^{\frac{1}{q}} \end{aligned}$$

$$+\frac{b-a}{4ab}\left[\int_0^1 tH_t^2\big(b,H(a,b)\big)\,dt\right]^{1-\frac{1}{q}}\left[\int_0^1 tH_t^2\big(b,H(a,b)\big)\left|f'\left(H_t\big(b,H(a,b)\big)\right)\right|^q dt\right]^{\frac{1}{q}}. \tag{3.2}$$

By Lemma 2.3, we have

$$\int_0^1 tH_t^2\big(a,H(a,b)\big)\,dt = S_1\big(a,H(a,b)\big) \tag{3.3}$$

and

$$\int_0^1 tH_t^2\big(b,H(a,b)\big)\,dt = S_1\big(b,H(a,b)\big). \tag{3.4}$$

Using the HT-convexity of f on $[a,b]$ and Lemma 2.3, we get

$$\begin{aligned}
&\int_0^1 tH_t^2\big(a,H(a,b)\big)\left|f'\left(H_t\big(a,H(a,b)\big)\right)\right|^q dt\\
\le&\int_0^1 tH_t^2\big(a,H(a,b)\big)\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(a)|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(H(a,b))|^q\right]dt\\
=&\frac{1}{2}\int_0^1\sqrt{t(1-t)^{-1}}H_t^2\big(a,H(a,b)\big)\left[t|f'(a)|^q+(1-t)|f'(H(a,b))|^q\right]dt\\
=&T\left(a,H(a,b);|f'(a)|^q,|f'(H(a,b))|^q\right)
\end{aligned} \tag{3.5}$$

and

$$\int_0^1 tH_t^2\big(b,H(a,b)\big)\left|f'\left(H_t\big(b,H(a,b)\big)\right)\right|^q dt \le T\left(b,H(a,b);|f'(a)|^q,|f'(H(a,b))|^q\right). \tag{3.6}$$

Applying identities (3.3) and (3.4) and inequalities (3.5) and (3.6) to (3.2) gives the inequality (3.1). The proof of Theorem 3.1 is thus completed. □

Corollary 3.2. *Let $I\subseteq\mathbb{R}\backslash\{0\}$ be an interval and $f:I\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, and $f'\in L_1([a,b])$. If $|f'|^q$ is an* HT*-convex function on $[a,b]$ for $q\ge 1$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx\right|\\
\le&\frac{b-a}{4ab}\left[S_1\left(|a|,H(|a|,|b|)\right)\right]^{1-\frac{1}{q}}\left[T\left(|a|,H(|a|,|b|);|f'(a)|^q,|f'(H(a,b))|^q\right)\right]^{\frac{1}{q}}\\
&+\frac{b-a}{4ab}\left[S_1(|b|,H(|a|,|b|))\right]^{1-\frac{1}{q}}\left[T\left(|b|,H(|a|,|b|);|f'(b)|^q,|f'(H(a,b))|^q\right)\right]^{\frac{1}{q}},
\end{aligned} \tag{3.7}$$

where $H(x,y)$ is the harmonic mean, $S_1(u,v)$ and $T\left(u,v;x,y\right)$ are defined as in Lemma 2.3.

Proof. Let $I\subseteq\mathbb{R}\backslash\{0\}$ be an interval and $a,b\in I^\circ$ with $a<b$. Consequently, we have $ab>0$ and

$$H_t^2\big(a,H(a,b)\big)=H_t^2\big(|a|,H(|a|,|b|)\big),\quad H_t^2\big(b,H(a,b)=H_t^2\big(|b|,H(|a|,|b|)\big).$$

Thus, from the identities (3.3) and (3.4) and the inequalities (3.2), (3.5) and (3.6), we get the inequality (3.7). Corollary 3.2 is thus proved. □

Theorem 3.3. *Let $f:I\subseteq\mathbb{R}_+\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, and $f'\in L_1([a,b])$. If $|f'|^q$ is an* HT*-convex function on $[a,b]$ for $q,p>1$ with $\frac{1}{q}+\frac{1}{p}=1$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx\right|\\
\le&\frac{b-a}{4ab}\left[S_p(a,H(a,b))\right]^{\frac{1}{p}}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\left[\frac{3|f'(a)|^q+|f'(H(a,b))|^q}{4}\right]^{\frac{1}{q}}\\
&+\frac{b-a}{4ab}\left[S_p(b,H(a,b))\right]^{\frac{1}{p}}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\left[\frac{3|f'(b)|^q+|f'(H(a,b))|^q}{4}\right]^{\frac{1}{q}},
\end{aligned} \tag{3.8}$$

where $H(x,y)$ is the harmonic mean and $S_p(u,v)$ is defined as in Lemma 2.3.

Proof. Using Lemma 2.1 and Hölder's integral inequality, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx\right| \\
&\le \frac{b-a}{4ab}\int_0^1\left[tH_t^2\big(b,H(a,b)\big)\left|f'\left(H_t\big(b,H(a,b)\big)\right)\right|+tH_t^2\big(a,H(a,b)\big)\left|f'\left(H_t\big(a,H(a,b)\big)\right)\right|\right]dt \\
&\le \frac{b-a}{4ab}\left[\int_0^1 tH_t^{2p}\big(a,H(a,b)\big)\,dt\right]^{\frac{1}{p}}\left[\int_0^1 t\left|f'\left(H_t\big(a,H(a,b)\big)\right)\right|^q dt\right]^{\frac{1}{q}} \\
&\quad+\frac{b-a}{4ab}\left[\int_0^1 tH_t^{2p}\big(b,H(a,b)\big)\,dt\right]^{\frac{1}{p}}\left[\int_0^1 t\left|f'\left(H_t\big(b,H(a,b)\big)\right)\right|^q dt\right]^{\frac{1}{q}}.
\end{aligned} \tag{3.9}$$

By Lemma 2.3 and the HT-convexity of f on $[a,b]$, we acquire

$$\int_0^1 tH_t^{2p}\big(a,H(a,b)\big)\,dt=S_p(a,H(a,b)), \tag{3.10}$$

$$\int_0^1 tH_t^{2p}\big(b,H(a,b)\big)\,dt=S_p(b,H(a,b)) \tag{3.11}$$

and

$$\begin{aligned}
\int_0^1 t\left|f'\left(H_t\big(a,H(a,b)\big)\right)\right|^q dt &\le \int_0^1 t\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(a)|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(H(a,b))|^q\right]dt \\
&=\frac{1}{16}\pi\left[3|f'(a)|^q+|f'(H(a,b))|^q\right],
\end{aligned} \tag{3.12}$$

$$\int_0^1 t\left|f'\left(H_t\big(b,H(a,b)\big)\right)\right|^q dt \le \frac{1}{16}\pi\left[3|f'(b)|^q+|f'(H(a,b))|^q\right]. \tag{3.13}$$

Using the identities (3.10) and (3.11) and the inequalities (3.12) and (3.13) to (3.9), the inequality (3.8) can be obtained. The proof of Theorem 3.3 is thus completed. □

Corollary 3.4. *Let $I\subseteq\mathbb{R}\backslash\{0\}$ be an interval and $f:I\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, and $f'\in L_1([a,b])$. If $|f'|^q$ is an* HT-*convex function on $[a,b]$ for $q,p>1$ with $\frac{1}{q}+\frac{1}{p}=1$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx\right| \\
&\le \frac{b-a}{4ab}\left[S_p(|a|,H(|a|,|b|))\right]^{\frac{1}{p}}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\left[\frac{3|f'(a)|^q+|f'(H(a,b))|^q}{4}\right]^{\frac{1}{q}} \\
&\quad+\frac{b-a}{4ab}\left[S_p(|b|,H(|a|,|b|))\right]^{\frac{1}{p}}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\left[\frac{3|f'(b)|^q+|f'(H(a,b))|^q}{4}\right]^{\frac{1}{q}},
\end{aligned}$$

where $H(x,y)$ is the harmonic mean and $S_p(u,v)$ is defined as in Lemma 2.3.

Theorem 3.5. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ with $a<b$, and $f' \in L_1([a,b])$. If $|f'|^q$ is an* HT*-convex function on $[a,b]$ for $q \geq 1$, then*

$$\left|\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx - f\left(\frac{2ab}{a+b}\right)\right| \leq \frac{b-a}{4ab}\left[S_1(H(a,b),a)\right]^{1-\frac{1}{q}}\left[T\left(H(a,b),a;|f'(H(a,b))|^q,|f'(a)|^q\right)\right]^{\frac{1}{q}} + \frac{b-a}{4ab}\left[S_1(H(a,b),b)\right]^{1-\frac{1}{q}}\left[T\left(H(a,b),b;|f'(H(a,b))|^q,|f'(b)|^q\right)\right]^{\frac{1}{q}}, \tag{3.14}$$

where $H(x,y)$ is the harmonic mean, $S_1(u,v)$ and $T\left(u,v;x,y\right)$ are defined as in Lemma 2.3.

Proof. From Lemma 2.2 and by replacing $\lambda = 1-t$ for $t \in [0,1]$ as an integral variable, we get

$$\begin{aligned}\left|\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx - f\left(\frac{2ab}{a+b}\right)\right| &\leq \frac{b-a}{4ab}\int_0^1 \left[(1-t)H_t^2(a,H(a,b))\left|f'\left(H_t(a,H(a,b))\right)\right| + (1-t)H_t^2(b,H(a,b))\left|f'\left(H_t(b,H(a,b))\right)\right|\right]dt \\ &= \frac{b-a}{4ab}\int_0^1 \left[\lambda H_\lambda^2(H(a,b),a)\left|f'\left(H_\lambda(H(a,b),a)\right)\right| + \lambda H_\lambda^2(H(a,b),b)\left|f'\left(H_\lambda(H(a,b),b)\right)\right|\right]d\lambda. \end{aligned} \tag{3.15}$$

Therefore, using (3.3) to (3.5) in the proof of Theorem 3.1, we complete the proof of Theorem 3.5. □

Corollary 3.6. *Let $I \subseteq \mathbb{R}\backslash\{0\}$ be an interval and $f : I \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ with $a<b$, and $f' \in L_1([a,b])$. If $|f'|^q$ is an* HT*-convex function on $[a,b]$ for $q \geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx\right| \leq \frac{b-a}{4ab}\left[S_1\left(H(|a|,|b|),|a|\right)\right]^{1-\frac{1}{q}}\left[T\left(H(|a|,|b|),|a|;|f'(H(a,b))|^q,|f'(a)|^q\right)\right]^{\frac{1}{q}} + \frac{b-a}{4ab}\left[S_1(H(|a|,|b|),|b|)\right]^{1-\frac{1}{q}}\left[T\left(H(|a|,|b|),|b|;|f'(H(a,b))|^q,|f'(b)|^q\right)\right]^{\frac{1}{q}},$$

where $H(x,y)$ is the harmonic mean, $S_1(u,v)$ and $T\left(u,v;x,y\right)$ are defined as in Lemma 2.3.

Theorem 3.7. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ with $a<b$, and $f' \in L_1([a,b])$. If $|f'|^q$ is an* HT*-convex function on $[a,b]$ for $q,p>1$ with $\frac{1}{q}+\frac{1}{p}=1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx\right| \leq \frac{b-a}{4ab}\left[S_p(H(a,b),a)\right]^{\frac{1}{p}}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\left[\frac{|f'(a)|^q+3|f'(H(a,b))|^q}{4}\right]^{\frac{1}{q}} + \frac{b-a}{4ab}\left[S_p(H(a,b),b)\right]^{\frac{1}{p}}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\left[\frac{|f'(b)|^q+3|f'(H(a,b))|^q}{4}\right]^{\frac{1}{q}},$$

where $H(x,y)$ is the harmonic mean and $S_p(u,v)$ is defined as in Lemma 2.3.

Proof. From the inequality (3.15) and Theorem 3.3, we obtain the proof of Theorem 3.7. □

Corollary 3.8. *Let $I \subseteq \mathbb{R}\backslash\{0\}$ be an interval and $f : I \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ with $a < b$, and $f' \in L_1([a,b])$. If $|f'|^q$ is an* HT-*convex function on $[a,b]$ for $q,p > 1$ with $\frac{1}{q}+\frac{1}{p} = 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx\right|$$
$$\leq \frac{b-a}{4ab}\left[S_p(H(|a|,|b|),|a|)\right]^{\frac{1}{p}}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\left[\frac{|f'(a)|^q+3|f'(H(a,b))|^q}{4}\right]^{\frac{1}{q}}$$
$$+\frac{b-a}{4ab}\left[S_p(H(|a|,|b|),|b|)\right]^{\frac{1}{p}}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\left[\frac{|f'(b)|^q+3|f'(H(a,b))|^q}{4}\right]^{\frac{1}{q}},$$

where $H(x,y)$ is the harmonic mean and $S_p(u,v)$ is defined as in Lemma 2.3.

4. Integral inequalities for product of MT-convex and HT-convex functions

Now we are in a position to establish some new integral inequalities for product of MT-convex and HT-convex functions

Theorem 4.1. *Let $I \subseteq \mathbb{R}_+$ be an interval and $f,g : I \to \mathbb{R}_+$. If the function f is* HT-*convex and g is* MT-*convex on I, and $fg \in L_1([a,b])$, where $a,b \in I^\circ$ with $a < b$, then*

$$f\big(H(a,b)\big)g\big(A(a,b)\big) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)g\left(\frac{ab}{x}\right)+f(x)g\left(a+b-\frac{ab}{x}\right)}{2x^2}\,dx \tag{4.1}$$

and

$$f\big(H(a,b)\big)g\big(A(a,b)\big) \leq \frac{1}{b-a}\int_a^b \frac{f\left(\frac{ab}{x}\right)g(x)+f\left(\frac{ab}{a+b-x}\right)g(x)}{2}\,dx, \tag{4.2}$$

where $A(u,v)$ is the arithmetic mean and $H(u,v)$ is the harmonic mean.

Proof. For all $t \in (0,1)$, we have

$$A(a,b) = A\big(A_t(a,b),A_{1-t}(a,b)\big),\;\; H(a,b) = H\big(H_t(a,b),H_{1-t}(a,b)\big)$$

and from the HT-convexity of f, we can see that

$$f\big(H(a,b)\big) = f\big(H\big(H_t(a,b),H_{1-t}(a,b)\big)\big) \leq A\left(f\big(H_t(a,b)\big),f\big(H_{1-t}(a,b)\big)\right),\;\; t \in (0,1) \tag{4.3}$$

and by the MT-convexity of g, we have

$$g\big(A(a,b)\big) = g\big(A\big(A_t(a,b),A_{1-t}(a,b)\big)\big) \leq A\left(g\big(A_t(a,b)\big),g\big(A_{1-t}(a,b)\big)\right),\;\; t \in (0,1). \tag{4.4}$$

Multiplying the two sides of inequality (4.3) and inequality (4.4), and finding the integral about t on both sides, we acquire

$$f\big(H(a,b)\big)g\big(A(a,b)\big) = \int_0^1 f\big(H(a,b)\big)g\big(A(a,b)\big)\,dt$$
$$\leq \int_0^1 A\left(f\big(H_t(a,b)\big),f\big(H_{1-t}(a,b)\big)\right)\cdot A\left(g\big(A_t(a,b)\big),g\big(A_{1-t}(a,b)\big)\right)\,dt \tag{4.5}$$
$$=\frac{1}{2}\int_0^1 \left[f\big(H_t(a,b)\big)g\big(A_t(a,b)\big)+f\big(H_t(a,b)\big)g\big(A_{1-t}(a,b)\big)\right]dt.$$

Using the variable change $x = H_t(a,b)$ for $t \in (0,1)$, then

$$A_t(a,b) = \frac{(a+b)x-ab}{x},\;\; A_{1-t}(a,b) = \frac{ab}{x}.$$

By the inequality (4.5), we can deduce that

$$\begin{aligned}
&f\big(H(a,b)\big)g\big(A(a,b)\big)\\
\leq&\frac{1}{2}\int_0^1\left[f\big(H_t(a,b)\big)g\big(A_t(a,b)\big)+f\big(H_t(a,b)\big)g\big(A_{1-t}(a,b)\big)\right]dt\\
=&\frac{ab}{b-a}\int_a^b\frac{f(x)g\left(\frac{ab}{x}\right)+f(x)g\left(a+b-\frac{ab}{x}\right)}{2x^2}\,dx.
\end{aligned}$$

Therefore, the inequality (4.1) holds.

Making an integral transformation $x=A_t(a,b)$ for $t\in(0,1)$, we obtain

$$H_t(a,b)=\frac{ab}{a+b-x},\quad H_{1-t}(a,b)=\frac{ab}{x}.$$

Using the inequality (4.5), we can get the inequality

$$\begin{aligned}
&f\big(H(a,b)\big)g\big(A(a,b)\big)\\
\leq&\frac{1}{2}\int_0^1\left[f\big(H_t(a,b)\big)g\big(A_t(a,b)\big)+f\big(H_{1-t}(a,b)\big)g\big(A_t(a,b)\big)\right]dt\\
=&\frac{1}{b-a}\int_a^b\frac{f\left(\frac{ab}{x}\right)g(x)+f\left(\frac{ab}{a+b-x}\right)g(x)}{2}\,dx.
\end{aligned}$$

The proof of Theorem 4.1 is complete. □

Corollary 4.2. *Let $I\subseteq\mathbb{R}_+$ be an interval and $f,g:I\to\mathbb{R}_+$. If the function f is* HT*-convex and g is* MT*-convex on I, and $fg\in L_1([a,b])$, where $a,b\in I^\circ$ with $a<b$, then*

1. *When g is a decreasing function on I, then*

$$f\big(H(a,b)\big)g\big(A(a,b)\big)\leq\frac{ab}{b-a}\int_a^b\frac{f(x)g(x)+f(x)g\left(\frac{ab}{x}\right)}{2x^2}\,dx; \tag{4.6}$$

2. *When f is an increasing function on I, then*

$$f\big(H(a,b)\big)g\big(A(a,b)\big)\leq\frac{1}{b-a}\int_a^b\frac{f(x)g(x)+f(x)g(a+b-x)}{2}\,dx, \tag{4.7}$$

where $A(u,v)$ is the arithmetic mean and $H(u,v)$ is the harmonic mean.

Proof. Since g is a decreasing function, we obtain $g(A_t(a,b))\leq g(H_t(a,b))$. By the inequality (4.5) in the proof of Theorem 4.1, so we have

$$\begin{aligned}
&f\big(H(a,b)\big)g\big(A(a,b)\big)\\
\leq&\frac{1}{2}\int_0^1\left[f\big(H_t(a,b)\big)g\big(A_t(a,b)\big)+f\big(H_t(a,b)\big)g\big(A_{1-t}(a,b)\big)\right]dt\\
\leq&\frac{1}{2}\int_0^1\left[f\big(H_t(a,b)\big)g\big(H_t(a,b)\big)+f\big(H_t(a,b)\big)g\big(H_{1-t}(a,b)\big)\right]dt.
\end{aligned} \tag{4.8}$$

Similarly, utilizing the monotonic increment of f, we can get

$$\begin{aligned}
&f\big(H(a,b)\big)g\big(A(a,b)\big)\\
\leq&\frac{1}{2}\int_0^1\left[f\big(H_t(a,b)\big)g\big(A_t(a,b)\big)+f\big(H_t(a,b)\big)g\big(A_{1-t}(a,b)\big)\right]dt\\
\leq&\frac{1}{2}\int_0^1\left[f\big(A_t(a,b)\big)g\big(A_t(a,b)\big)+f\big(A_t(a,b)\big)g\big(A_{1-t}(a,b)\big)\right]dt.
\end{aligned} \tag{4.9}$$

Consequently, the inequalities (4.6) and (4.7) can be deduced from the inequalities (4.8) and (4.9). Corollary 4.2 is thus proved. □

Theorem 4.3. *Let $I \subseteq \mathbb{R}_+$ be an interval and $f,g : I \to \mathbb{R}_+$. If the function f is* HT*-convex and g is* MT*-convex on I, and $fg \in L_1([a,b])$, where $a,b \in I^\circ$ with $a<b$, then*

$$\frac{ab}{b-a}\int_a^b \tau_H(x)\frac{f(x)g(x)}{x^2}\,dx \le \frac{\pi}{32\sqrt{ab}}\big[bg(a)\big(3f(a)+f(b)\big)+ag(b)\big(f(a)+3f(b)\big)\big] \tag{4.10}$$

and

$$\frac{1}{b-a}\int_a^b \tau_A(x)f(x)g(x)\,dx \le \frac{\pi}{32\sqrt{ab}}\big[af(a)\big(3g(a)+g(b)\big)+bf(b)\big(g(a)+3g(b)\big)\big], \tag{4.11}$$

where

$$\tau_H(x)=\frac{\sqrt{ab(x-a)(b-x)}}{(b-a)x},\quad \tau_A(x)=\frac{\sqrt{(x-a)(b-x)}}{b-a},\quad x\in[a,b].$$

Proof. By the variable change $x=H_t(a,b)$ for $t\in(0,1)$, then $\tau_H(x)=\sqrt{t(1-t)}$ for $t\in(0,1)$. Also by the HT-convexity of f, we gain

$$\frac{ab}{b-a}\int_a^b \tau_H(x)\frac{f(x)g(x)}{x^2}\,dx=\int_0^1\sqrt{t(1-t)}f\big(H_t(a,b)\big)g\big(H_t(a,b)\big)\,dt\le\frac{1}{2}\int_0^1 A_t\big(f(a),f(b)\big)g\big(H_t(a,b)\big)\,dt.$$

Changing the integral variable again and using the MT-convexity of g, we obtain

$$\begin{aligned}
&\frac{ab}{b-a}\int_a^b \tau_H(x)\frac{f(x)g(x)}{x^2}\,dx\\
\le&\frac{1}{2}\int_0^1 A_t\big(f(a),f(b)\big)g\big(H_t(a,b)\big)\,dt\\
=&\frac{1}{2}\int_0^1 \frac{taf(a)+(1-t)bf(b)}{A_t(a,b)}g\big(A_t(a,b)\big)\frac{ab}{[A_t(a,b)]^2}\,dt\\
\le&\frac{ab}{4}\int_0^1\frac{[taf(a)+(1-t)bf(b)]\,[tg(a)+(1-t)g(b)]}{\sqrt{t(1-t)}[A_t(a,b)]^3}\,dt\\
=&\frac{\pi}{32\sqrt{ab}}\big[bg(a)\big(3f(a)+f(b)\big)+ag(b)\big(f(a)+3f(b)\big)\big].
\end{aligned}$$

Similarly, we can obtain the second inequality (4.11). Thus the proof of Theorem 4.3 is completed. □

Corollary 4.4. *Let $I \subseteq \mathbb{R}_+$ be an interval and $f,g : I \to \mathbb{R}_+$. If the function f is* HT*-convex and g is* MT*-convex on I, and $fg \in L_1([a,b])$, where $a,b \in I^\circ$ with $a<b$.*

1. *When g is an increasing function on I, then*

$$\frac{ab}{b-a}\int_a^b \tau_H(x)\frac{f(x)g(x)}{x^2}\,dx \le \frac{\pi}{32}\big[f(a)(3g(a)+g(b))+f(b)(g(a)+3g(b))\big];$$

2. *When f is a decreasing function on I, then*

$$\frac{1}{b-a}\int_a^b \tau_A(x)f(x)g(x)\,dx \le \frac{\pi}{32}\big[f(a)(3g(a)+g(b))+f(b)(g(a)+3g(b))\big],$$

where $\tau_H(x)$ and $\tau_A(x)$ are defined by Theorem 4.3.

Proof. Since g is an increasing function on I, by the inequality (4.10), and the HT-convexity of function f and the MT-convexity of function g, we deduced that

$$\begin{aligned}&\frac{ab}{b-a}\int_a^b \tau_H(x)\frac{f(x)g(x)}{x^2}\,dx\\ \le&\int_0^1 \sqrt{t(1-t)}f\big(H_t(a,b)\big)g\big(A_t(a,b)\big)\,dt\\ \le&\frac{1}{4}\int_0^1 \frac{1}{\sqrt{t(1-t)}}A_t\big(f(a),f(b)\big)A_t\big(g(a),g(b)\big)\,dt\\ =&\frac{\pi}{32}\big[f(a)(3g(a)+g(b))+f(b)(g(a)+3g(b))\big]\end{aligned}$$

and utilizing the decreasing of function f, we acquire

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \tau_A(x)f(x)g(x)\,dx\\ \le&\int_0^1 \sqrt{t(1-t)}f\big(H_t(a,b)\big)g\big(A_t(a,b)\big)\,dt\\ \le&\frac{\pi}{32}\big[f(a)(3g(a)+g(b))+f(b)(g(a)+3g(b))\big].\end{aligned}$$

The proof of Corollary 4.4 is completed. □

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Use of AI Tools

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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