



Research Article

A Unified Proof of Three Combinatorial Identities Related to the Stirling Numbers of the Second Kind

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Abstract

In the note, the authors give a unified proof of Identities 67, 84, and 85 in the monograph “M. Z. Spivey, *The Art of Proving Binomial Identities*, Discrete Mathematics and its Applications, CRC Press, Boca Raton, FL, 2019; available online at <https://doi.org/10.1201/9781351215824>” and connect these three identities with a computing formula for the Stirling numbers of the second kind. Moreover, in terms of the notion of Qi’s normalized remainders of the exponential and logarithmic functions, the authors reformulate the definitions of the Stirling numbers of the first and second kind and their generalizations by Howard in 1967 and 1980, Carlitz in 1980, and Broder in 1984.

Keywords: Combinatorial identity, unified proof, Stirling number, normalized remainder, reformulation

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1. Introduction

Let $m, n \in \mathbb{N}_0$ be nonnegative integers.

Identity 12 on [1, p. 12] is

$$\sum_{k=0}^n \binom{n}{k} (-1)^k = \begin{cases} 1, & n = 0; \\ 0, & n \in \mathbb{N}. \end{cases} \quad (1)$$

Identity 84 on [1, p. 62] states

$$\sum_{k=0}^n \binom{n}{k} (n-k)^m (-1)^k = 0, \quad m < n. \quad (2)$$

Identity 67 on [1, p. 55] reads that

$$\sum_{k=0}^n \binom{n}{k} (n-k)^n (-1)^k = n!. \quad (3)$$

Identity 85 on [1, p. 62] gives

$$\sum_{k=0}^n \binom{n}{k} (n-k)^{n+1} (-1)^k = \frac{n(n+1)!}{2}. \quad (4)$$

It is easy to see that the second case $n \in \mathbb{N}$ in (1) is a special case of (2). These identities can also be found in the monographs [2, 3].

The only main aim of this note is to provide a unified proof for the above three identities (2) to (4).

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2. Original proof of the identity (3)

We recite the original proof on [1, p. 68] of the identity (3) as follows.

By the binomial theorem, we have

$$\sum_{k=0}^n \binom{n}{k} e^{(n-k)x} (-1)^k = (e^x - 1)^n. \quad (5)$$

Now, differentiate both sides n times. The left side is

$$\sum_{k=0}^n \binom{n}{k} (n-k)^n e^{(n-k)x} (-1)^k.$$

The right side is (after a derivative or two)

$$\frac{d^{n-1}}{dx^{n-1}} [n(e^x - 1)^{n-1} e^x] = \frac{d^{n-2}}{dx^{n-2}} [n(n-1)(e^x - 1)^{n-2} e^{2x} + n(e^x - 1)^{n-1} e^x].$$

After n derivatives we will have n terms. One of these will be the term $n!e^{nx}$. All of the others will contain at least one factor of $e^x - 1$. Now, let $x = 0$. The $e^{(n-k)x}$ factor vanishes on the left side, and everything on the right goes to zero except for $n!e^0$. The identity follows.

3. A unified proof of three identities

It is well known [4, p. 51] that the Stirling numbers of the second kind $S(k, n)$ for $k \geq n \in \mathbb{N}_0$ can be analytically generated by

$$\frac{(e^x - 1)^n}{n!} = \sum_{k=n}^{\infty} S(k, n) \frac{x^k}{k!}. \quad (6)$$

Hence, we obtain

$$(e^x - 1)^n = \sum_{k=0}^{\infty} \frac{S(k+n, n)}{\binom{k+n}{n}} \frac{x^{k+n}}{k!}, \quad n \in \mathbb{N}_0. \quad (7)$$

Substituting the Maclaurin power series expansion (7) into (5) leads to

$$\sum_{k=0}^n \binom{n}{k} e^{(n-k)x} (-1)^k = \sum_{k=0}^{\infty} \frac{S(k+n, n)}{\binom{k+n}{n}} \frac{x^{k+n}}{k!}, \quad n \in \mathbb{N}_0. \quad (8)$$

Differentiating m times with respect to x on both sides of (8) results in

$$\sum_{k=0}^n \binom{n}{k} (n-k)^m e^{(n-k)x} (-1)^k = \sum_{k=0}^{\infty} \frac{S(k+n, n)}{\binom{k+n}{n}} (k+n)_m \frac{x^{k+n-m}}{k!} \quad (9)$$

for $m, n \in \mathbb{N}_0$, where

$$\langle z \rangle_n = \prod_{k=0}^{n-1} (z-k) = \begin{cases} z(z-1) \cdots (z-n+1), & n \in \mathbb{N} \\ 1, & n = 0 \end{cases}$$

is the falling factorial of $z \in \mathbb{C}$.

Taking the limit $x \rightarrow 0$ on both sides of (9) reveals

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} (n-k)^m (-1)^k &= \begin{cases} \frac{S(m, n)}{\binom{m}{n}} \frac{\langle m \rangle_m}{(m-n)!}, & m \geq n \in \mathbb{N}_0 \\ 0, & n > m \in \mathbb{N}_0 \end{cases} \\ &= \begin{cases} n! S(m, n), & m \geq n \in \mathbb{N}_0 \\ 0, & n > m \in \mathbb{N}_0 \end{cases} \end{aligned} \quad (10)$$

for $m, n \in \mathbb{N}_0$.

The case $n > m \in \mathbb{N}_0$ in (10) is just the identity (2).

Letting $m = n$ in (10) gives the identity (3).

Taking $m = n + 1$ in (10) and employing $S(n + 1, n) = \frac{n(n+1)}{2}$ for $n \in \mathbb{N}_0$, we easily derive the identity (4). The unified proof is complete.

4. Remarks

Finally, we list several remarks.

Remark 1. The unified proof in Section 3 is more understandable and comprehensive than the original proof recited in Section 2 of the identity (3).

Remark 2. We can write the equality (10) as the formula

$$S(m, n) = \begin{cases} \frac{1}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)^m, & m \geq n \in \mathbb{N}_0; \\ 0, & n > m \in \mathbb{N}_0, \end{cases} \quad (11)$$

see [1, p. 193, Identity 224]. Consequently, since $S(n, n) = 1$ for $n \in \mathbb{N}_0$, the identity (3) is the special case $m = n$ of the formula (11), which can be rearranged as the form

$$S(m, n) = \begin{cases} \frac{(-1)^n}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} k^m, & m > n \in \mathbb{N}_0; \\ 1, & m = n \in \mathbb{N}_0; \\ 0, & n > m \in \mathbb{N}_0, \end{cases} \quad (12)$$

see [4, p. 204, Theorem A].

Remark 3. The three identities (2), (3), and (4) can be reformulated as

$$\sum_{k=0}^n \binom{n}{k} k^m (-1)^k = 0, \quad m < n,$$

$$\sum_{k=0}^n \binom{n}{k} k^n (-1)^k = (-1)^n n!,$$

and

$$\sum_{k=0}^n \binom{n}{k} k^{n+1} (-1)^k = (-1)^n \frac{n(n+1)!}{2}.$$

Remark 4. The Maclaurin power series expansion (6) can be reformulated as

$$\left(\frac{e^x - 1}{x} \right)^n = \sum_{k=0}^{\infty} \frac{S(k+n, n)}{\binom{k+n}{n}} \frac{x^k}{k!}, \quad n \geq 0. \quad (13)$$

The Bernoulli numbers B_k for $k \geq 0$ are generated by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{x^{2k}}{(2k)!}, \quad |x| < 2\pi, \quad (14)$$

see [4, p. 48]. Comparing the generating functions in (13) and (14), considering the fact that $\frac{e^x - 1}{x}$ and $\frac{x}{e^x - 1}$ are the reciprocal of each other, we are sure that the Bernoulli numbers B_k and the Stirling numbers of the second kind $S(n, k)$

must have something to do with each other. This idea has been carried out and verified by Qi and his coauthors in several papers.

On the other hand, the generating functions $\left(\frac{e^x-1}{x}\right)^n$ and $\frac{x}{e^x-1}$ of the Stirling numbers of the second kind $S(n, k)$ and the Bernoulli numbers B_k have been generalized in [5–7] by

$$(T_r[e^x])^\ell = \frac{\ell![(r+1)!]^\ell}{[(r+1)\ell]!} \sum_{j=0}^{\infty} \frac{S_r(j+(r+1)\ell, \ell)}{\binom{j+(r+1)\ell}{j}} \frac{x^j}{j!} \tag{15}$$

and

$$\frac{e^{xt}}{T_{r-1}[e^x]} = \sum_{j=0}^{\infty} A_{r,j}(t) \frac{x^j}{j!}$$

respectively, where

$$T_r[e^x] = \frac{(r+1)!}{x^{r+1}} \left(e^x - \sum_{j=0}^r \frac{x^j}{j!} \right), \quad r \in \mathbb{N}_0 \tag{16}$$

is called Qi’s normalized remainder of the exponential function e^x in the literature [8, 9].

Remark 5. In [6], Howard defined $s_r(j, \ell)$ by

$$\left[\ln \frac{1}{1-x} - \sum_{j=1}^r \frac{x^j}{j} \right]^\ell = \ell! \sum_{j=(r+1)\ell}^{\infty} s_r(j, \ell) \frac{x^j}{j!}, \quad r \in \mathbb{N}_0. \tag{17}$$

It is clear that $s_0(j, \ell) = (-1)^{j+\ell} s(j, \ell)$, where the Stirling numbers of the first kind $s(j, \ell)$ can be analytically generated [10, Theorem 3.14] by

$$\left[\frac{\ln(1+x)}{x} \right]^\ell = \sum_{j=0}^{\infty} \frac{s(j+\ell, \ell)}{\binom{j+\ell}{\ell}} \frac{x^j}{j!}, \quad |x| < 1. \tag{18}$$

The equation (17) can be reformulated as

$$\left[(-1)^r \frac{r+1}{x^r} \left(\frac{\ln(1+x)}{x} - \sum_{j=0}^{r-1} (-1)^j \frac{x^j}{j+1} \right) \right]^\ell = \frac{\ell!(r+1)^\ell}{[(r+1)\ell]!} \sum_{j=0}^{\infty} (-1)^j \frac{s_r(j+(r+1)\ell, \ell)}{\binom{j+(r+1)\ell}{j}} \frac{x^j}{j!} \tag{19}$$

for $r \in \mathbb{N}_0$. The function

$$(-1)^r \frac{r+1}{x^r} \left[\frac{\ln(1+x)}{x} - \sum_{j=0}^{r-1} (-1)^j \frac{x^j}{j+1} \right] \tag{20}$$

in (19) is just Qi’s normalized remainder $T_r\left[\frac{\ln(1+x)}{x}\right]$ for $r \in \mathbb{N}_0$ of the function $\frac{\ln(1+x)}{x}$.

Remark 6. Theorem 15 in [11] reads that the r -Stirling numbers of the first kind for $r \in \mathbb{N}_0$ have the “vertical” exponential generating function

$$\sum_k \begin{bmatrix} k+r \\ m+r \end{bmatrix}_r \frac{z^k}{k!} = \begin{cases} \frac{1}{m!} \left(\frac{1}{1-z}\right)^r \left(\ln \frac{1}{1-z}\right)^m, & m \geq 0; \\ 0, & m < 0. \end{cases} \tag{21}$$

We can rewrite (21) in the form

$$\left(\frac{1}{1+z}\right)^r \left[\frac{\ln(1+z)}{z}\right]^m = \left(\frac{1}{1+z}\right)^r \left(T_0\left[\frac{\ln(1+z)}{z}\right]\right)^m = \sum_{k=0}^{\infty} (-1)^k \frac{\begin{bmatrix} k+m+r \\ m+r \end{bmatrix}_r z^k}{\binom{k+m}{m} k!}, \quad r, m \in \mathbb{N}_0, \quad |z| < 1. \tag{22}$$

Taking $r = 0$ in (19) and (22) and comparing with (18) give

$$\begin{bmatrix} k \\ m \end{bmatrix}_0 = s_0(k, m) = (-1)^{k+m} s(k, m), \quad k, m \in \mathbb{N}_0.$$

Basing on the above discussion, we propose a problem: Investigate the properties of the sequence $F(r, s, m, k)$ generated by

$$\left(\frac{1}{1+z}\right)^r \left(T_s \left[\frac{\ln(1+z)}{z}\right]\right)^m = \sum_{k=0}^{\infty} F(r, s, m, k) \frac{z^k}{k!}, \quad r, m \in \mathbb{C}, \quad s \in \mathbb{N}_0, \quad |z| < 1. \tag{23}$$

Remark 7. Theorem 16 in [11] and Eq. (3.9) in [12] state that the r -Stirling numbers of the second kind for $r \in \mathbb{N}_0$ have the exponential generating function

$$\sum_k \begin{Bmatrix} k+r \\ m+r \end{Bmatrix}_r \frac{z^k}{k!} = \begin{cases} \frac{1}{m!} e^{rz} (e^z - 1)^m, & m \geq 0; \\ 0, & m < 0. \end{cases} \tag{24}$$

We can reformulate (24) in the form

$$e^{rz} (T_0[e^z])^m = \sum_{k=0}^{\infty} \frac{\begin{Bmatrix} k+m+r \\ m+r \end{Bmatrix}_r z^k}{\binom{k+m}{m} k!}, \quad r, m \in \mathbb{N}_0, \quad |z| < \infty. \tag{25}$$

Taking $r = 0$ in (15) and (25) and comparing with (13) yield

$$\begin{Bmatrix} k \\ m \end{Bmatrix}_0 = S_0(k, m) = S(k, m), \quad k, m \in \mathbb{N}_0.$$

Basing on the above discussion, we propose a problem: Investigate the properties of the sequence $Q(r, s, m, k)$ generated by

$$e^{rz} (T_s[e^z])^m = \sum_{k=0}^{\infty} Q(r, s, m, k) \frac{z^k}{k!}, \quad r, m \in \mathbb{C}, \quad s \in \mathbb{N}_0?$$

Remark 8. We recall from the papers [8, 9, 13–15] that Qi’s normalized remainder can be generally defined as follows.

Let f be a real infinitely differentiable function on an interval I such that $0 \in I \subseteq \mathbb{R}$. If $f^{(n+1)}(0) \neq 0$ for some $n \in \mathbb{N}_0$, then the function

$$T_n[f(x)] = \begin{cases} \frac{1}{f^{(n+1)}(0)} \frac{(n+1)!}{x^{n+1}} \left[f(x) - \sum_{k=0}^n f^{(k)}(0) \frac{x^k}{k!} \right], & x \neq 0 \\ 1, & x = 0 \end{cases} \tag{26}$$

for $x \in I$ is said to be the n th normalized remainder or the n th normalized tail of the Maclaurin expansion of the function f .

Applying $f(x)$ in (26) to e^x leads to the normalized remainder $T_n[e^x]$ for $n \in \mathbb{N}_0$ defined by (16), replacing $f(x)$ by $\frac{\ln(1+x)}{x}$ in (26) reduces to the normalized remainder $T_n\left[\frac{\ln(1+x)}{x}\right]$ for $n \in \mathbb{N}_0$ defined by (20).

It is easy to verify that $T_n\left[\frac{\ln(1+x)}{x}\right] = T_n[\ln(1+x)]$ for $n \in \mathbb{N}_0$. Therefore, the equations (19) and (23) can be reformulated respectively by

$$(T_r[\ln(1+x)])^\ell = \frac{\ell!(r+1)^\ell}{[(r+1)\ell]!} \sum_{j=0}^{\infty} (-1)^j \frac{s_r(j+(r+1)\ell, \ell) x^j}{\binom{j+(r+1)\ell}{j} j!}, \quad r, \ell \in \mathbb{N}_0, \quad |x| < 1$$

and

$$\left(\frac{1}{1+z}\right)^r (T_s[\ln(1+z)])^m = \sum_{k=0}^{\infty} F(r, s, m, k) \frac{z^k}{k!}, \quad r, m \in \mathbb{C}, \quad s \in \mathbb{N}_0, \quad |z| < 1.$$

Remark 9. This paper is a condensed version of the preprint arXiv:2509.09753 at the site <https://doi.org/10.48550/arXiv.2509.09753>.

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Use of AI Tools

AI tools were not employed in generating, analyzing, or interpreting the results.

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