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Research Article

On Second-Family Radau-type Inequalities

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Abstract

In this paper, we establish a new integral identity and derive novel Radau-type integral inequalities for functions whose first derivatives are convex. These results extend and generalize some existing inequalities of Radau type in the literature. As an application, we provide bounds for certain mathematical means based on the obtained inequalities. We believe that the findings presented here will stimulate further research into integral inequalities and their applications in related fields.

Keywords: Radau-type inequalities, convex functions, Hölder inequality, Young inequality, power mean inequality

2020 MSC: 26D10, 26A51, 26D15.

1. Introduction

A function $\mathfrak{h} : I \subset \mathbb{R} \to \mathbb{R}$ is deemed to be convex, if

$$\mathfrak{h}\left(\mu\iota_1 + (1-\mu)\iota_2\right) \le \mu\mathfrak{h}(\iota_1) + (1-\mu)\mathfrak{h}(\iota_2)$$

holds for all $\iota_1, \iota_2 \in I$ and all $\mu \in [0,1]$.

Convexity theory plays a central role in both pure and applied mathematics due to its rich geometric and algebraic structure, as well as its deep connections with various fields such as optimization, functional analysis, and mathematical inequalities [1–3]. One of its most notable features is its strong and intrinsic relationship with inequality theory; see [4] for an in-depth discussion. Among the many inequalities associated with convex functions, the Hermite–Hadamard inequality stands out as one of the most significant and widely studied. This fundamental result can be stated as follows:

Let $\mathfrak{h} : [\iota_1, \iota_2] \to \mathbb{R}$ be a convex function. Then,

$$\mathfrak{h}\left(\frac{\iota_1+\iota_2}{2}\right) \le \frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)du \le \frac{\mathfrak{h}(\iota_1)+\mathfrak{h}(\iota_2)}{2}. \tag{1.1}$$

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This double inequality provides bounds for the average value of a convex function over an interval and has served as a cornerstone for numerous generalizations and extensions in modern mathematical analysis [5–8].

The concept of convexity has been extensively studied in the literature to derive error estimates for various quadrature formulas, such as the midpoint [9, 10], trapezium [11, 12], Simpson [13, 14], Bullen [12, 15], among others. This underscores the importance and utility of convexity in establishing integral inequalities.

The approximation of definite integrals is a central task in many areas of mathematics, physics, and engineering, especially when closed-form expressions are not available or difficult to compute. To address this challenge, various quadrature rules have been developed over the years, each offering different trade-offs between simplicity, accuracy, and computational efficiency.

Among the most classical and widely used methods are the Newton–Cotes formulas, which approximate the integral of a function by evaluating it at equally spaced points within the interval of integration. These include well-known rules such as the trapezoidal rule, Simpson's rule, among others. While easy to implement and intuitive, Newton–Cotes formulas often suffer from limited accuracy unless a large number of evaluation points is used, and they can become unstable for high–degree polynomial approximations due to Runge's phenomenon.

In contrast, Gaussian quadrature formulas offer a more sophisticated approach by selecting both the nodes and the weights in an optimal way. Specifically, Gaussian rules are designed to integrate exactly all polynomials of degree up to $2n-1$ using only n nodes, making them significantly more accurate than Newton–Cotes formulas for the same number of function evaluations. The nodes in Gaussian quadrature are not uniformly distributed but are instead chosen as the roots of orthogonal polynomials on the given interval.

A particularly useful variant of Gaussian quadrature arises in the case where one endpoint of the interval is fixed as a node. This leads to the so–called Radau quadrature rules, which provide a balance between the flexibility of full Gaussian rules and the boundary inclusion typical of Newton–Cotes methods. Radau–type inequalities, in particular, have found applications in error estimation, numerical analysis, and the derivation of bounds for integral approximations under regularity assumptions on the integrand, such as convexity or boundness.

The work [16] by Rebai and Meftah introduces a 2-point right Radau-type inequality via convexity.

Theorem 1.1 ([16]). *Let $\mathfrak{h}:[\iota_1,\iota_2]\to\mathbb{R}$ be a differentiable function on $[\iota_1,\iota_2]$ with $\iota_1<\iota_2$. If $|\mathfrak{h}'|$ is convex, then the following inequality holds*

$$\left|\frac{3}{4}\mathfrak{h}\left(\tfrac{2\iota_1+\iota_2}{3}\right)+\frac{1}{4}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_1}\mathfrak{h}(u)du\right|\leq\frac{\iota_2-\iota_1}{18}\left(\frac{1}{3}|\mathfrak{h}(\iota_1)|+\frac{379}{192}\left|\mathfrak{h}\left(\tfrac{2\iota_1+\iota_2}{3}\right)\right|+\frac{157}{192}|\mathfrak{h}(\iota_2)|\right).$$

Remark 1.2. By using the convexity of $|\mathfrak{h}'|$, i.e: $\left|\mathfrak{h}'\left(\frac{2\iota_1+\iota_2}{3}\right)\right|\leq\frac{2|\mathfrak{h}'(\iota_1)|+|\mathfrak{h}'(\iota_2)|}{3}$, Theorem 1.1 yields

$$\left|\frac{3}{4}\mathfrak{h}\left(\tfrac{2\iota_1+\iota_2}{3}\right)+\frac{1}{4}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right|\leq\frac{\iota_2-\iota_1}{5184}\left(475\left|\mathfrak{h}'(\iota_1)\right|+425\left|\mathfrak{h}'(\iota_2)\right|\right).$$

Within the same theoretical framework, Meftah et al. [17] introduced a 2-point left Radau-type inequality tailored for differentiable convex mappings, given by:

Theorem 1.3 ([17]). *Let $\mathfrak{h}:[\iota_1,\iota_2]\to\mathbb{R}$ be a differentiable function on $[\iota_1,\iota_2]$ with $\iota_1<\iota_2$. If $|\mathfrak{h}'|$ is convex, then the following inequality holds*

$$\left|\frac{1}{4}\left(\mathfrak{h}(\iota_1)+\frac{3}{4}\mathfrak{h}\left(\tfrac{\iota_1+2\iota_2}{3}\right)\right)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)du\right|\leq\frac{\iota_2-\iota_1}{18}\left(\frac{157|\mathfrak{h}'(\iota_1)|+379\left|\mathfrak{h}'\left(\tfrac{\iota_1+2\iota_2}{3}\right)\right|+64|\mathfrak{h}'(\iota_2)|}{192}\right).$$

Remark 1.4. By using the convexity of $|\mathfrak{h}'|$, i.e: $\left|\mathfrak{h}'\left(\frac{\iota_1+2\iota_2}{3}\right)\right|\leq\frac{|\mathfrak{h}'(\iota_1)|+2|\mathfrak{h}'(\iota_2)|}{3}$, Theorem 1.3 yields

$$\left|\frac{1}{4}\mathfrak{h}(\iota_1)+\frac{3}{4}\mathfrak{h}\left(\tfrac{\iota_1+2\iota_2}{3}\right)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right|\leq\frac{\iota_2-\iota_1}{5184}\left(425\left|\mathfrak{h}'(\iota_1)\right|+475\left|\mathfrak{h}'(\iota_2)\right|\right).$$

In recent years, several extensions of 2-point Radau-type inequalities have been established in generalized settings, reflecting the growing interest in refining integral approximation techniques under various analytical frameworks. In the context of fractional calculus, Liu et al. [18] derived error bounds for the 2-point right Radau formula under s-convexity assumptions. On the other hand, in the framework of multiplicative calculus, Nasri et al. [19] investigated left-Radau-type inequalities for multiplicative differentiable s-convex functions.

In [20], the authors provided the following second family Radau-type inequalities.

Theorem 1.5 ([20])**.** *Let $\mathfrak{h}:[-1,1]\to\mathbb{R}$ be a function satisfying $\mathfrak{h}''\in L_\infty[-1,1]$. Then, for $x\in(-1,-\frac{1}{3}]\cup[\frac{1}{3},1)$, we have*

$$\left|\int_{-1}^{1}\mathfrak{h}(u)\,du-\frac{1+3x}{3(1+x)}\mathfrak{h}(-1)-\frac{4}{3(1-x^2)}\mathfrak{h}(x)-\frac{1-3x}{3(1-x)}\mathfrak{h}(1)\right|\leq\frac{4}{81}\left(\frac{1+3|x|}{1+|x|}\right)^3\left\|\mathfrak{h}''\right\|_\infty.$$

Building upon the works presented in [16, 17], and in pursuit of a deeper understanding of error bounds for Radau-type quadrature formulas, we investigate in this work second-family Radau-type inequalities for functions whose first derivatives are convex. As a consequence of our analysis, several established results from the literature are also derived as special cases.

The remainder of this paper is structured as follows: In Section 2, we recall a known result that will be used throughout our analysis, and we introduce a new integral identity that serves as the foundation for deriving our main results. Section 3 presents several new Radau-type inequalities of the second family under convexity assumptions on the first derivative. From these results, various previously established inequalities are recovered as special cases. To validate the precision and illustrate the applicability of our findings, Section 4 provides a set of illustrative examples accompanied by graphical representations. Section 5 is devoted to applications involving special means, where we derive bounds based on the obtained inequalities. Finally, concluding remarks are given in Section 6.

2. Auxiliary results

This section provides a brief overview of a known result that will be instrumental in our analysis and introduces a new integral identity that forms the basis for deriving our main inequalities.

Lemma 2.1 ([21])**.** *Let $\mathfrak{h}:[\imath_1,\imath_2]\to(0,\infty)$, if $\mathfrak{h}^q$ is convex on $[\imath_1,\imath_2]$ for all $q\in(0,1]$, we can derive two important consequences below.*

1. *For $0<q\leq\frac{1}{2}$, we have*

$$\begin{aligned}&\mathfrak{h}((1-\mu)\imath_1+\mu\imath_2)\\ \leq& q2^{\frac{1}{q}-1}\left((1-\mu)^{\frac{1}{q}}\mathfrak{h}(\imath_1)+\mu^{\frac{1}{q}}\mathfrak{h}(\imath_2)+\left(\tfrac{2}{q}-2\right)\mu^{\frac{1}{2q}}(1-\mu)^{\frac{1}{2q}}(\mathfrak{h}(\imath_1)\mathfrak{h}(\imath_2))^{\frac{1}{2}}\right).\end{aligned}\tag{2.1}$$

2. *For $\frac{1}{2}<q\leq1$, we have*

$$\begin{aligned}&\mathfrak{h}((1-\mu)\imath_1+\mu\imath_2)\\ \leq&(1-\mu)^{\frac{1}{q}}\mathfrak{h}(\imath_1)+\mu^{\frac{1}{q}}\mathfrak{h}(\imath_2)+\left(2^{\frac{1}{q}}-2\right)\mu^{\frac{1}{2q}}(1-\mu)^{\frac{1}{2q}}(\mathfrak{h}(\imath_1)\mathfrak{h}(\imath_2))^{\frac{1}{2}}.\end{aligned}\tag{2.2}$$

The following lemma is required to support our findings.

Lemma 2.2. *Let $\mathfrak{h}:[\imath_1,\imath_2]\to\mathbb{R}$ be a differentiable function on $[\imath_1,\imath_2]$ with $\imath_1<\imath_2$, and $\mathfrak{h}'\in L^1[\imath_1,\imath_2]$, then the following equality holds*

$$\frac{3\lambda-1}{6\lambda}\mathfrak{h}(\imath_1)+\frac{1}{6\lambda(1-\lambda)}\mathfrak{h}((1-\lambda)\imath_1+\lambda\imath_2)+\frac{2-3\lambda}{6(1-\lambda)}\mathfrak{h}(\imath_2)-\frac{1}{\imath_2-\imath_1}\int_{\imath_1}^{\imath_2}\mathfrak{h}(u)\,du$$

$$=(\iota_2-\iota_1)\left(\int_0^\lambda\left(\mu-\frac{3\lambda-1}{6\lambda}\right)\mathfrak{h}'((1-\mu)\iota_1+\mu\iota_2)\,d\mu+\int_\lambda^1\left(\mu-\frac{4-3\lambda}{6(1-\lambda)}\right)\mathfrak{h}'((1-\mu)\iota_1+\mu\iota_2)\,d\mu\right),$$

where $\lambda\in\left(0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right)$.

Proof. Let

$$\mathscr{O}_1=\int_0^\lambda\left(\mu-\frac{3\lambda-1}{6\lambda}\right)\mathfrak{h}'((1-\mu)\iota_1+\mu\iota_2)\,d\mu$$

and

$$\mathscr{O}_2=\int_\lambda^1\left(\mu-\frac{4-3\lambda}{6(1-\lambda)}\right)\mathfrak{h}'((1-\mu)\iota_1+\mu\iota_2)\,d\mu.$$

Integrating by parts $\mathscr{O}_1$, we obtain

$$\begin{aligned}\mathscr{O}_1&=\int_0^\lambda\left(\mu-\frac{3\lambda-1}{6\lambda}\right)\mathfrak{h}'((1-\mu)\iota_1+\mu\iota_2)\,d\mu\\&=\frac{1}{\iota_2-\iota_1}\left(\mu-\frac{3\lambda-1}{6\lambda}\right)\mathfrak{h}((1-\mu)\iota_1+\mu\iota_2)\Big|_0^\lambda-\frac{1}{\iota_2-\iota_1}\int_0^\lambda\mathfrak{h}((1-\mu)\iota_1+\mu\iota_2)\,d\mu\\&=\frac{1}{\iota_2-\iota_1}\left(\lambda-\tfrac{3\lambda-1}{6\lambda}\right)\mathfrak{h}((1-\lambda)\iota_1+\lambda\iota_2)+\tfrac{1}{\iota_2-\iota_1}\left(\tfrac{3\lambda-1}{6\lambda}\right)\mathfrak{h}(\iota_1)-\tfrac{1}{(\iota_2-\iota_1)^2}\int_{\iota_1}^{(1-\lambda)\iota_1+\lambda\iota_2}\mathfrak{h}(u)\,du.\end{aligned}\tag{2.3}$$

Similarly, we obtain

$$\begin{aligned}\mathscr{O}_2&=\int_\lambda^1\left(\mu-\frac{4-3\lambda}{6(1-\lambda)}\right)\mathfrak{h}'((1-\mu)\iota_1+\mu\iota_2)\,d\mu\\&=\frac{1}{\iota_2-\iota_1}\left(\mu-\frac{4-3\lambda}{6(1-\lambda)}\right)\mathfrak{h}((1-\mu)\iota_1+\mu\iota_2)\Big|_\lambda^1-\frac{1}{\iota_2-\iota_1}\int_\lambda^1\mathfrak{h}((1-\mu)\iota_1+\mu\iota_2)\,d\mu\\&=\frac{1}{\iota_2-\iota_1}\left(1-\tfrac{4-3\lambda}{6(1-\lambda)}\right)\mathfrak{h}(\iota_2)-\tfrac{1}{\iota_2-\iota_1}\left(\lambda-\tfrac{4-3\lambda}{6(1-\lambda)}\right)\mathfrak{h}((1-\lambda)\iota_1+\lambda\iota_2)-\tfrac{1}{(\iota_2-\iota_1)^2}\int_{\lambda\iota_1+(1-\lambda)\iota_2}^{\iota_2}\mathfrak{h}(u)\,du.\end{aligned}\tag{2.4}$$

Summing (2.3) and (2.4), then multiplying the resulting equality by $(\iota_2-\iota_1)$, we get the desired result. □

3. Main results

Building on the identity introduced in the previous section, we derive several new Radau-type integral inequalities of the second family under convexity assumptions on the first derivative.

Theorem 3.1. *Let* $\mathfrak{h}$ *be as in Lemma 2.2. If* $|\mathfrak{h}'|$ *is convex, then we have*

$$\begin{aligned}&\left|\frac{3\lambda-1}{6\lambda}\mathfrak{h}(\iota_1)+\frac{1}{6\lambda(1-\lambda)}\mathfrak{h}((1-\lambda)\iota_1+\lambda\iota_2)+\frac{2-3\lambda}{6(1-\lambda)}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right|\\&\le(\iota_2-\iota_1)\left[(\Lambda_1(\lambda)+\Lambda_2(\lambda))\left|\mathfrak{h}'(\iota_1)\right|+(\Lambda_3(\lambda)+\Lambda_4(\lambda))\left|\mathfrak{h}'(\iota_2)\right|\right],\end{aligned}$$

where $\Lambda_i(\lambda)$, $i=1,\ldots,4$ *are defined as in* (3.1)-(3.4)*, respectively.*

Proof. From Lemma 2.2, properties of absolute value, and convexity of $|\mathfrak{h}'|$, we have

$$\left|\frac{3\lambda-1}{6\lambda}\mathfrak{h}(\iota_1)+\frac{1}{6\lambda(1-\lambda)}\mathfrak{h}((1-\lambda)\iota_1+\lambda\iota_2)+\frac{2-3\lambda}{6(1-\lambda)}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right|$$

$$\leq(\iota_2-\iota_1)\left[\int_0^\lambda\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|\left|\mathfrak{h}'((1-\mu)\iota_1+\mu\iota_2)\right|d\mu+\int_\lambda^1\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|\left|\mathfrak{h}'((1-\mu)\iota_1+\mu\iota_2)\right|d\mu\right]$$

$$\leq(\iota_2-\iota_1)\left[\int_0^\lambda\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|\left((1-\mu)\left|\mathfrak{h}'(\iota_1)\right|+\mu\left|\mathfrak{h}'(\iota_2)\right|\right)d\mu+\int_\lambda^1\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|\left((1-\mu)\left|\mathfrak{h}'(\iota_1)\right|+\mu\left|\mathfrak{h}'(\iota_2)\right|\right)d\mu\right]$$

$$=(\iota_2-\iota_1)\Bigg[\left|\mathfrak{h}'(\iota_1)\right|\int_0^\lambda\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|(1-\mu)\,d\mu+\left|\mathfrak{h}'(\iota_2)\right|\int_0^\lambda\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|\mu d\mu$$

$$+\left|\mathfrak{h}'(\iota_1)\right|\int_\lambda^1\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|(1-\mu)\,d\mu+\left|\mathfrak{h}'(\iota_2)\right|\int_\lambda^1\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|\mu d\mu\Bigg]$$

$$=(\iota_2-\iota_1)\Bigg[\left(\int_0^\lambda\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|(1-\mu)\,d\mu+\int_\lambda^1\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|(1-\mu)\,d\mu\right)\left|\mathfrak{h}'(\iota_1)\right|$$

$$+\left(\int_0^\lambda\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|\mu d\mu+\int_\lambda^1\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|\mu d\mu\right)\left|\mathfrak{h}'(\iota_2)\right|\Bigg]$$

$$=(\iota_2-\iota_1)\left[(\Lambda_1(\lambda)+\Lambda_2(\lambda))\left|\mathfrak{h}'(\iota_1)\right|+(\Lambda_3(\lambda)+\Lambda_4(\lambda))\left|\mathfrak{h}'(\iota_2)\right|\right],$$

where we have used

$$\Lambda_1(\lambda)=\int_0^\lambda\left|\mu-\frac{3\lambda-1}{6\lambda}\right|(1-\mu)\,d\mu=\begin{cases}\frac{2-7\lambda+9\lambda^2-4\lambda^3}{12} & \text{if } \lambda\leq\frac{1}{3},\\ \frac{2-7\lambda+9\lambda^2-4\lambda^3}{12}+\frac{1+15\lambda}{18\lambda}\left(\frac{3\lambda-1}{6\lambda}\right)^2 & \text{if } \lambda>\frac{1}{3},\end{cases} \tag{3.1}$$

$$\Lambda_2(\lambda)=\int_\lambda^1\left|\mu-\frac{4-3\lambda}{6(1-\lambda)}\right|(1-\mu)\,d\mu=\begin{cases}\frac{14-15\lambda}{18(1-\lambda)}\left(\frac{4-3\lambda}{6(1-\lambda)}\right)^2+\frac{-2-7\lambda+16\lambda^2-13\lambda^3+4\lambda^4}{12(1-\lambda)} & \text{if } \lambda\leq\frac{2}{3},\\ \frac{2-9\lambda+16\lambda^2-13\lambda^3+4\lambda^4}{12(1-\lambda)} & \text{if } \lambda>\frac{2}{3},\end{cases} \tag{3.2}$$

$$\Lambda_3(\lambda)=\int_0^\lambda\left|\mu-\frac{3\lambda-1}{6\lambda}\right|\mu d\mu=\begin{cases}\frac{\lambda-3\lambda^2+4\lambda^3}{12} & \text{if } \lambda\leq\frac{1}{3},\\ \frac{\lambda-3\lambda^2+4\lambda^3}{12}+\frac{1}{3}\left(\frac{3\lambda-1}{6\lambda}\right)^3 & \text{if } \lambda>\frac{1}{3}\end{cases} \tag{3.3}$$

and

$$\Lambda_4(\lambda)=\int_\lambda^1\left|\mu-\frac{4-3\lambda}{6(1-\lambda)}\right|\mu d\mu=\begin{cases}\frac{1}{3}\left(\frac{4-3\lambda}{6(1-\lambda)}\right)^3-\frac{\lambda+4\lambda^2-7\lambda^3+4\lambda^4}{12(1-\lambda)} & \text{if } \lambda\leq\frac{2}{3},\\ \frac{\lambda-4\lambda^2+7\lambda^3-4\lambda^4}{12(1-\lambda)} & \text{if } \lambda>\frac{2}{3}.\end{cases} \tag{3.4}$$

The proof is achieved. □

Remark 3.2. Theorem 3.1 provides a general framework that generates a variety of specific results depending on the choice of the parameter λ. Indeed, by selecting different values of λ, one can recover several known inequalities as particular cases. Among these, we highlight the following:

1. For $\lambda=\frac{1}{3}$, Theorem 3.1 coincides with Remark 1.2.
2. For $\lambda=\frac{2}{3}$, Theorem 3.1 aligns with Remark 1.4.

This flexibility highlights the richness and unifying nature of the proposed estimate.

Theorem 3.3. *Let $\mathfrak{h}$ be as in Lemma 2.2. If $|\mathfrak{h}'|^q$ is convex where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$\left|\frac{3\lambda-1}{6\lambda}\mathfrak{h}(\iota_1)+\frac{1}{6\lambda(1-\lambda)}\mathfrak{h}((1-\lambda)\iota_1+\lambda\iota_2)+\frac{2-3\lambda}{6(1-\lambda)}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right|$$
$$\leq(\iota_2-\iota_1)\left((\Xi_1(\lambda,p))^{\frac{1}{p}}\left(\tfrac{2\lambda-\lambda^2}{2}\left|\mathfrak{h}'(\iota_1)\right|^q+\tfrac{\lambda^2}{2}\left|\mathfrak{h}'(\iota_2)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+(\Xi_2(\lambda,p))^{\frac{1}{p}}\left(\tfrac{(1-\lambda)^2}{2}\left|\mathfrak{h}'(\iota_1)\right|^q+\tfrac{1-\lambda^2}{2}\left|\mathfrak{h}'(\iota_2)\right|^q\right)^{\frac{1}{q}}\right),$$

where $\Xi_1(\lambda,p)$ and $\Xi_2(\lambda,p)$ are defined by (3.5) and (3.6), respectively.

Proof. From Lemma 2.2, properties of absolute value, Hölder's inequality, and convexity of $|\mathfrak{h}'|^q$, it yields

$$\left|\frac{3\lambda-1}{6\lambda}\mathfrak{h}(\iota_1)+\frac{1}{6\lambda(1-\lambda)}\mathfrak{h}((1-\lambda)\iota_1+\lambda\iota_2)+\frac{2-3\lambda}{6(1-\lambda)}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right|$$
$$\leq(\iota_2-\iota_1)\left(\left(\int_0^\lambda\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^\lambda\left|\mathfrak{h}'((1-\mu)\iota_1+\mu\iota_2)\right|^q d\mu\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_\lambda^1\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_\lambda^1\left|\mathfrak{h}'((1-\mu)\iota_1+\mu\iota_2)\right|^q d\mu\right)^{\frac{1}{q}}\right)$$
$$\leq(\iota_2-\iota_1)\left(\left(\int_0^\lambda\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^\lambda\left((1-\mu)\left|\mathfrak{h}'(\iota_1)\right|^q+\mu\left|\mathfrak{h}'(\iota_2)\right|^q\right)d\mu\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_\lambda^1\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_\lambda^1\left((1-\mu)\left|\mathfrak{h}'(\iota_1)\right|^q+\mu\left|\mathfrak{h}'(\iota_2)\right|^q\right)d\mu\right)^{\frac{1}{q}}\right)$$
$$=(\iota_2-\iota_1)\left((\Xi_1(\lambda,p))^{\frac{1}{p}}\left(\tfrac{2\lambda-\lambda^2}{2}\left|\mathfrak{h}'(\iota_1)\right|^q+\tfrac{\lambda^2}{2}\left|\mathfrak{h}'(\iota_2)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+(\Xi_2(\lambda,p))^{\frac{1}{p}}\left(\tfrac{(1-\lambda)^2}{2}\left|\mathfrak{h}'(\iota_1)\right|^q+\tfrac{1-\lambda^2}{2}\left|\mathfrak{h}'(\iota_2)\right|^q\right)^{\frac{1}{q}}\right),$$

where we have used

$$\Xi_1(\lambda,p)=\int_0^\lambda\left|\mu-\frac{3\lambda-1}{6\lambda}\right|^p d\mu=\begin{cases}\frac{\left(6\lambda^2-3\lambda+1\right)^{p+1}-(1-3\lambda)^{p+1}}{6^{p+1}\lambda^{p+1}(p+1)} & \text{if } \lambda\leq\frac{1}{3},\\ \frac{(3\lambda-1)^{p+1}+\left(6\lambda^2-3\lambda+1\right)^{p+1}}{6^{p+1}\lambda^{p+1}(p+1)} & \text{if } \lambda>\frac{1}{3}\end{cases} \tag{3.5}$$

and

$$\Xi_2(\lambda,p)=\int_{\lambda}^{1}\left|\mu-\frac{4-3\lambda}{6(1-\lambda)}\right|^p d\mu=\begin{cases}\frac{\left(4-9\lambda+6\lambda^2\right)^{p+1}+(2-3\lambda)^{p+1}}{6^{p+1}(1-\lambda)^{p+1}(p+1)} & \text{if } \lambda\le\frac{2}{3},\\ \frac{\left(4-9\lambda+6\lambda^2\right)^{p+1}-(3\lambda-2)^{p+1}}{6^{p+1}(1-\lambda)^{p+1}(p+1)} & \text{if } \lambda>\frac{2}{3}.\end{cases} \tag{3.6}$$

The proof is finished. □

Corollary 3.4. *If we choose $\lambda=\frac{1}{3}$ in Theorem 3.3, then we obtain the following 2-point right Radau-type inequality*

$$\begin{aligned}&\left|\frac{3}{4}\mathfrak{h}\left(\tfrac{2\iota_1+\iota_2}{3}\right)+\frac{1}{4}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right|\\ &\le\frac{\iota_2-\iota_1}{18}\left(\tfrac{1}{p+1}\right)^{\frac{1}{p}}\left(2\left(\tfrac{5|\mathfrak{h}'(\iota_1)|^q+|\mathfrak{h}'(\iota_2)|^q}{6}\right)^{\frac{1}{q}}+\left(\tfrac{5^{p+1}+3^{p+1}}{8}\right)^{\frac{1}{p}}\left(\tfrac{|\mathfrak{h}'(\iota_1)|^q+2|\mathfrak{h}'(\iota_2)|^q}{3}\right)^{\frac{1}{q}}\right),\end{aligned}$$

which coincides with the result obtained in Corollary 2 from [16], where the convexity of $|\mathfrak{h}'|$ was applied to the term $\left|\mathfrak{h}'\left(\frac{2\iota_1+\iota_2}{3}\right)\right|$.

Corollary 3.5. *Choosing $\lambda=\frac{2}{3}$, Theorem 3.3 yields the following 2-point left Radau-type inequality*

$$\begin{aligned}&\left|\frac{1}{4}\mathfrak{h}(\iota_1)+\frac{3}{4}\mathfrak{h}\left(\tfrac{\iota_1+2\iota_2}{3}\right)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right|\\ &\le\frac{\iota_2-\iota_1}{18}\left(\tfrac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(\tfrac{3^{p+1}+5^{p+1}}{8}\right)^{\frac{1}{p}}\left(\tfrac{2|\mathfrak{h}'(\iota_1)|^q+|\mathfrak{h}'(\iota_2)|^q}{3}\right)^{\frac{1}{q}}+2\left(\tfrac{|\mathfrak{h}'(\iota_1)|^q+5|\mathfrak{h}'(\iota_2)|^q}{6}\right)^{\frac{1}{q}}\right),\end{aligned}$$

which aligns with the one established in Corollary 2 of [17], where the convexity of $|\mathfrak{h}'|$ was applied to estimate the term $\left|\mathfrak{h}'\left(\frac{\iota_1+2\iota_2}{3}\right)\right|$.

Theorem 3.6. *Let $\mathfrak{h}$ be as in Lemma 2.2. If $|\mathfrak{h}'|^q$ is convex where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$\begin{aligned}&\left|\frac{3\lambda-1}{6\lambda}\mathfrak{h}(\iota_1)+\frac{1}{6\lambda(1-\lambda)}\mathfrak{h}((1-\lambda)\iota_1+\lambda\iota_2)+\frac{2-3\lambda}{6(1-\lambda)}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right|\\ &\le(\iota_2-\iota_1)\left((\Xi_1(\lambda,p))^{\frac{1}{p}}\left(\tfrac{\lambda}{2}\right)^{\frac{1}{q}}\left(\left|\mathfrak{h}'(\iota_1)\right|^q+\left|\mathfrak{h}'((1-\lambda)\iota_1+\lambda\iota_2)\right|^q\right)^{\frac{1}{q}}\right.\\ &\quad\left.+(\Xi_2(\lambda,p))^{\frac{1}{p}}\left(\tfrac{1-\lambda}{2}\right)^{\frac{1}{q}}\left(\left|\mathfrak{h}'((1-\lambda)\iota_1+\lambda\iota_2)\right|^q+\left|\mathfrak{h}'(\iota_2)\right|^q\right)^{\frac{1}{q}}\right),\end{aligned}$$

where $\Xi_1(\lambda,p)$ and $\Xi_2(\lambda,p)$ are defined by (3.5) and (3.6), respectively.

Proof. From Lemma 2.2, properties of absolute value, Hölder's inequality, and convexity of $|\mathfrak{h}'|^q$, it yields

$$\begin{aligned}&\left|\frac{3\lambda-1}{6\lambda}\mathfrak{h}(\iota_1)+\frac{1}{6\lambda(1-\lambda)}\mathfrak{h}((1-\lambda)\iota_1+\lambda\iota_2)+\frac{2-3\lambda}{6(1-\lambda)}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right|\\ &\le(\iota_2-\iota_1)\left(\left(\int_0^{\lambda}\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^{\lambda}\left|\mathfrak{h}'((1-\mu)\iota_1+\mu\iota_2)\right|^q d\mu\right)^{\frac{1}{q}}\right.\end{aligned}$$

$$+\left(\int_{\lambda}^{1}\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|^{p}d\mu\right)^{\frac{1}{p}}\left(\int_{\lambda}^{1}\left|\mathfrak{h}'\left((1-\mu)\iota_1+\mu\iota_2\right)\right|^{q}d\mu\right)^{\frac{1}{q}}\Bigg)$$

$$=(\iota_2-\iota_1)\left(\left(\Xi_1(\lambda,p)\right)^{\frac{1}{p}}\left(\int_{0}^{\lambda}\left|\mathfrak{h}'\left((1-\mu)\iota_1+\mu\iota_2\right)\right|^{q}d\mu\right)^{\frac{1}{q}}+\left(\Xi_2(\lambda,p)\right)^{\frac{1}{p}}\left(\int_{\lambda}^{1}\left|\mathfrak{h}'\left((1-\mu)\iota_1+\mu\iota_2\right)\right|^{q}d\mu\right)^{\frac{1}{q}}\right). \tag{3.7}$$

Since $|\mathfrak{h}'|^q$ is convex, from Hermite-Hadamard inequality, we have

$$\int_{0}^{\lambda}\left|\mathfrak{h}'\left((1-\mu)\iota_1+\mu\iota_2\right)\right|^{q}d\mu\leq\lambda\frac{|\mathfrak{h}'(\iota_1)|^q+|\mathfrak{h}'((1-\lambda)\iota_1+\lambda\iota_2)|^q}{2} \tag{3.8}$$

and

$$\int_{\lambda}^{1}\left|\mathfrak{h}'\left((1-\mu)\iota_1+\mu\iota_2\right)\right|^{q}d\mu\leq(1-\lambda)\frac{|\mathfrak{h}'((1-\lambda)\iota_1+\lambda\iota_2)|^q+|\mathfrak{h}'(\iota_2)|^q}{2}. \tag{3.9}$$

Using (3.8) and (3.9) in (3.7), we get the desired result. □

Theorem 3.7. *Let $\mathfrak{h}$ be as in Lemma 2.2. If $|\mathfrak{h}'|^q$ is convex, where $q\geq 1$, then we have*

$$\left|\frac{3\lambda-1}{6\lambda}\mathfrak{h}(\iota_1)+\frac{1}{6\lambda(1-\lambda)}\mathfrak{h}((1-\lambda)\iota_1+\lambda\iota_2)+\frac{2-3\lambda}{6(1-\lambda)}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right|$$

$$\leq(\iota_2-\iota_1)\left(\Xi_1(\lambda,p)+\Xi_2(\lambda,p)\right)^{\frac{1}{p}}\left(\tfrac{|\mathfrak{h}'(\iota_1)|^q+|\mathfrak{h}'(\iota_2)|^q}{2}\right)^{\frac{1}{q}},$$

where $\Xi_1(\lambda,p)$ and $\Xi_2(\lambda,p)$ are defined by (3.5) and (3.6), respectively.

Proof. Clearly, the identity of Lemma 2.2 can be rewritten as follows:

$$\frac{3\lambda-1}{6\lambda}\mathfrak{h}(\iota_1)+\frac{1}{6\lambda(1-\lambda)}\mathfrak{h}((1-\lambda)\iota_1+\lambda\iota_2)+\frac{2-3\lambda}{6(1-\lambda)}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du$$

$$=(\iota_2-\iota_1)\int_{0}^{1}K(\mu,\lambda)\,\mathfrak{h}((1-\mu)\iota_1+\mu\iota_2)\,d\mu, \tag{3.10}$$

with

$$K(\mu,\lambda)=\begin{cases}\mu-\frac{3\lambda-1}{6\lambda} & \text{if} \quad \mu\in[0,\lambda]\\ \mu-\frac{4-3\lambda}{6(1-\lambda)} & \text{if} \quad \mu\in[\lambda,1].\end{cases}$$

Using the absolute value on both sides of (3.10), then applying Hölder's inequality, we get

$$\left|\frac{3\lambda-1}{6\lambda}\mathfrak{h}(\iota_1)+\frac{1}{6\lambda(1-\lambda)}\mathfrak{h}((1-\lambda)\iota_1+\lambda\iota_2)+\frac{2-3\lambda}{6(1-\lambda)}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right|$$

$$\leq(\iota_2-\iota_1)\left(\int_{0}^{1}|K(\mu,\lambda)|^p\,d\mu\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\mathfrak{h}'\left((1-\mu)\iota_1+\mu\iota_2\right)\right|^{q}d\mu\right)^{\frac{1}{q}}$$

$$= (\iota_2-\iota_1)\left(\int_0^{\lambda}\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|^p d\mu+\int_{\lambda}^{1}\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathfrak{h}'\left((1-\mu)\iota_1+\mu\iota_2\right)\right|^q d\mu\right)^{\frac{1}{q}}$$

$$= (\iota_2-\iota_1)\left(\Xi_1(\lambda,p)+\Xi_2(\lambda,p)\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathfrak{h}'\left((1-\mu)\iota_1+\mu\iota_2\right)\right|^q d\mu\right)^{\frac{1}{q}}. \tag{3.11}$$

Since $|\mathfrak{h}'|^q$ is convex, from Hermite-Hadamard inequality, we have

$$\int_0^1\left|\mathfrak{h}'\left((1-\mu)\iota_1+\mu\iota_2\right)\right|^q d\mu\le\frac{|\mathfrak{h}'(\iota_1)|^q+|\mathfrak{h}'(\iota_2)|^q}{2}. \tag{3.12}$$

Using (3.12) in (3.11) we get the desired result. □

Theorem 3.8. *Let $\mathfrak{h}$ be as in Lemma 2.2. If $|\mathfrak{h}'|^q$ is convex, where $q\ge 1$, then we have*

$$\left|\frac{3\lambda-1}{6\lambda}\mathfrak{h}(\iota_1)+\frac{1}{6\lambda(1-\lambda)}\mathfrak{h}\left((1-\lambda)\iota_1+\lambda\iota_2\right)+\frac{2-3\lambda}{6(1-\lambda)}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right|$$

$$\le(\iota_2-\iota_1)\Bigg(\left(\Sigma_1(\lambda)\right)^{1-\frac{1}{q}}\left(\Lambda_1(\lambda)|\mathfrak{h}'(\iota_1)|^q+\Lambda_3(\lambda)|\mathfrak{h}'(\iota_2)|^q\right)^{\frac{1}{q}}$$

$$+\left(\Sigma_2(\lambda)\right)^{1-\frac{1}{q}}\left(\Lambda_2(\lambda)|\mathfrak{h}'(\iota_1)|^q+\Lambda_4(\lambda)|\mathfrak{h}'(\iota_2)|^q\right)^{\frac{1}{q}}\Bigg),$$

where $\Lambda_i(\lambda)$ $i=1$ to 4 and $\Sigma_1(\lambda)$ and $\Sigma_2(\lambda)$ are defined by (3.1)*-*(3.4) *and* (3.13) *and* (3.13)*, respectively.*

Proof. From Lemma 2.2, power mean inequality along with the convexity of $|\mathfrak{h}'|^q$, it yields

$$\left|\frac{3\lambda-1}{6\lambda}\mathfrak{h}(\iota_1)+\frac{1}{6\lambda(1-\lambda)}\mathfrak{h}\left((1-\lambda)\iota_1+\lambda\iota_2\right)+\frac{2-3\lambda}{6(1-\lambda)}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right|$$

$$\le(\iota_2-\iota_1)\Bigg(\left(\int_0^{\lambda}\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|d\mu\right)^{1-\frac{1}{q}}\left(\int_0^{\lambda}\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|\left|\mathfrak{h}'\left((1-\mu)\iota_1+\mu\iota_2\right)\right|^q d\mu\right)^{\frac{1}{q}}$$

$$+\left(\int_{\lambda}^{1}\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|d\mu\right)^{1-\frac{1}{q}}\left(\int_{\lambda}^{1}\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|\left|\mathfrak{h}'\left((1-\mu)\iota_1+\mu\iota_2\right)\right|^q d\mu\right)^{\frac{1}{q}}\Bigg)$$

$$\le(\iota_2-\iota_1)\Bigg(\left(\int_0^{\lambda}\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|d\mu\right)^{1-\frac{1}{q}}\left(\int_0^{\lambda}\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|\left((1-\mu)|\mathfrak{h}'(\iota_1)|^q+\mu|\mathfrak{h}'(\iota_2)|^q\right)d\mu\right)^{\frac{1}{q}}$$

$$+\left(\int_{\lambda}^{1}\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|d\mu\right)^{1-\frac{1}{q}}\left(\int_{\lambda}^{1}\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|\left((1-\mu)|\mathfrak{h}'(\iota_1)|^q+\mu|\mathfrak{h}'(\iota_2)|^q\right)d\mu\right)^{\frac{1}{q}}\Bigg)$$

$$=(\iota_2-\iota_1)\Bigg(\left(\int_0^{\lambda}\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|d\mu\right)^{1-\frac{1}{q}}\left(|\mathfrak{h}'(\iota_1)|^q\int_0^{\lambda}\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|(1-\mu)\,d\mu+|\mathfrak{h}'(\iota_2)|^q\int_0^{\lambda}\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|\mu\,d\mu\right)^{\frac{1}{q}}$$

$$+\left(\int\limits_{\lambda}^{1}\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|d\mu\right)^{1-\frac{1}{q}}\left(\left|\mathfrak{h}'(\iota_1)\right|^q\int\limits_{\lambda}^{1}\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|(1-\mu)\,d\mu+\left|\mathfrak{h}'(\iota_2)\right|^q\int\limits_{\lambda}^{1}\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|\mu d\mu\right)^{\frac{1}{q}}\Bigg)$$

$$=(\iota_2-\iota_1)\Big((\Sigma_1(\lambda))^{1-\frac{1}{q}}\left(\Lambda_1(\lambda)\left|\mathfrak{h}'(\iota_1)\right|^q+\Lambda_3(\lambda)\left|\mathfrak{h}'(\iota_2)\right|^q\right)^{\frac{1}{q}}$$

$$+(\Sigma_2(\lambda))^{1-\frac{1}{q}}\left(\Lambda_2(\lambda)\left|\mathfrak{h}'(\iota_1)\right|^q+\Lambda_4(\lambda)\left|\mathfrak{h}'(\iota_2)\right|^q\right)^{\frac{1}{q}}\Big),$$

where we have used (3.1)-(3.4), and the facts that

$$\Sigma_1(\lambda)=\int\limits_0^{\lambda}\left|\mu-\frac{3\lambda-1}{6\lambda}\right|d\mu=\begin{cases}\frac{1-3\lambda+3\lambda^2}{6} & \text{if} \quad \lambda\leq\frac{1}{3},\\ \frac{1-3\lambda+3\lambda^2}{6}+\left(\frac{3\lambda-1}{6\lambda}\right)^2 & \text{if} \quad \lambda>\frac{1}{3},\end{cases}$$

and

$$\Sigma_2(\lambda)=\int\limits_{\lambda}^{1}\left|\mu-\frac{4-3\lambda}{6(1-\lambda)}\right|d\mu=\begin{cases}\left(\frac{4-3\lambda}{6(1-\lambda)}\right)^2-\frac{1+4\lambda-6\lambda^2+3\lambda^3}{6(1-\lambda)} & \text{if} \quad \lambda\leq\frac{2}{3},\\ \frac{1-3\lambda+3\lambda^2}{6} & \text{if} \quad \lambda>\frac{2}{3}.\end{cases}$$

The proof is achieved. □

Theorem 3.9. *Let $\mathfrak{h}$ be as in Lemma 2.2. If $|\mathfrak{h}'|^q$ is convex, where $0<q\leq\frac{1}{2}$, then we have*

$$\left|\frac{3\lambda-1}{6\lambda}\mathfrak{h}(\iota_1)+\frac{1}{6\lambda(1-\lambda)}\mathfrak{h}((1-\lambda)\iota_1+\lambda\iota_2)+\frac{2-3\lambda}{6(1-\lambda)}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int\limits_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right|$$

$$\leq\frac{q(\iota_2-\iota_1)}{2^{1-\frac{1}{q}}}\Big((\Omega_1(\lambda)+\Omega_2(\lambda))\left|\mathfrak{h}'(\iota_1)\right|+(\Omega_3(\lambda)+\Omega_4(\lambda))\left|\mathfrak{h}'(\iota_2)\right|$$

$$+\left(\tfrac{2}{q}-2\right)(\Omega_5(\lambda)+\Omega_6(\lambda))\left(\left|\mathfrak{h}'(\iota_1)\right|\left|\mathfrak{h}'(\iota_2)\right|\right)^{\frac{1}{2}}\Big),$$

where $\Omega_i(\lambda)$ for $i=1$ to 6 are defined by (3.14)-(3.19), respectively.

Proof. Using the absolute value on both sides of the equality in Lemma 2.2 allows us to get

$$\left|\frac{3\lambda-1}{6\lambda}\mathfrak{h}(\iota_1)+\frac{1}{6\lambda(1-\lambda)}\mathfrak{h}((1-\lambda)\iota_1+\lambda\iota_2)+\frac{2-3\lambda}{6(1-\lambda)}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int\limits_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right|$$

$$\leq(\iota_2-\iota_1)\left(\int\limits_0^{\lambda}\left|\left(\mu-\tfrac{3\lambda-1}{6\lambda}\right)\right|\left|\mathfrak{h}'((1-\mu)\iota_1+\mu\iota_2)\right|d\mu+\int\limits_{\lambda}^{1}\left|\left(\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right)\right|\left|\mathfrak{h}'((1-\mu)\iota_1+\mu\iota_2)\right|d\mu\right). \tag{3.13}$$

Since, $|\mathfrak{h}'|^q$ is convex with $0<q\leq\frac{1}{2}$, we have from Lemma 2.1

$$\left|\frac{3\lambda-1}{6\lambda}\mathfrak{h}(\iota_1)+\frac{1}{6\lambda(1-\lambda)}\mathfrak{h}((1-\lambda)\iota_1+\lambda\iota_2)+\frac{2-3\lambda}{6(1-\lambda)}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int\limits_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right|$$

$$\leq q2^{\frac{1}{q}-1}(\iota_2-\iota_1)\left(\int\limits_0^{\lambda}\left|\left(\mu-\tfrac{3\lambda-1}{6\lambda}\right)\right|\left((1-\mu)^{\frac{1}{q}}\left|\mathfrak{h}'(\iota_1)\right|+\mu^{\frac{1}{q}}\left|\mathfrak{h}'(\iota_2)\right|+\left(\tfrac{2}{q}-2\right)\mu^{\frac{1}{2q}}(1-\mu)^{\frac{1}{2q}}\left(\left|\mathfrak{h}'(\iota_1)\right|\left|\mathfrak{h}'(\iota_2)\right|\right)^{\frac{1}{2}}\right)d\mu\right.$$

$$+\int_{\lambda}^{1}\left|\left(\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right)\right|\left((1-\mu)^{\frac{1}{q}}\left|\mathfrak{h}'(\iota_1)\right|+\mu^{\frac{1}{q}}\left|\mathfrak{h}'(\iota_2)\right|+\left(\tfrac{2}{q}-2\right)\mu^{\frac{1}{2q}}(1-\mu)^{\frac{1}{2q}}\left(\left|\mathfrak{h}'(\iota_1)\right|\left|\mathfrak{h}'(\iota_2)\right|\right)^{\frac{1}{2}}\right)d\mu\right)$$

$$=\frac{q(\iota_2-\iota_1)}{2^{1-\frac{1}{q}}}\left(\left|\mathfrak{h}'(\iota_1)\right|\left(\int_0^{\lambda}\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|(1-\mu)^{\frac{1}{q}}d\mu+\int_{\lambda}^{1}\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|(1-\mu)^{\frac{1}{q}}d\mu\right)\right.$$

$$+\left|\mathfrak{h}'(\iota_2)\right|\left(\int_0^{\lambda}\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|\mu^{\frac{1}{q}}d\mu+\int_{\lambda}^{1}\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|\mu^{\frac{1}{q}}d\mu\right)$$

$$\left.+\left(\tfrac{2}{q}-2\right)\left(\left|\mathfrak{h}'(\iota_1)\right|\left|\mathfrak{h}'(\iota_2)\right|\right)^{\frac{1}{2}}\left(\int_0^{\lambda}\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|\mu^{\frac{1}{2q}}(1-\mu)^{\frac{1}{2q}}d\mu+\int_{\lambda}^{1}\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|\mu^{\frac{1}{2q}}(1-\mu)^{\frac{1}{2q}}d\mu\right)\right)$$

$$=\frac{q(\iota_2-\iota_1)}{2^{1-\frac{1}{q}}}\Big((\Omega_1(\lambda)+\Omega_2(\lambda))\left|\mathfrak{h}'(\iota_1)\right|+(\Omega_3(\lambda)+\Omega_4(\lambda))\left|\mathfrak{h}'(\iota_2)\right|$$

$$+\left(\tfrac{2}{q}-2\right)(\Omega_5(\lambda)+\Omega_6(\lambda))\left(\left|\mathfrak{h}'(\iota_1)\right|\left|\mathfrak{h}'(\iota_2)\right|\right)^{\frac{1}{2}}\Big),$$

where we have used

$$\Omega_1(\lambda)=\int_0^{\lambda}\left|\mu-\frac{3\lambda-1}{6\lambda}\right|(1-\mu)^{\frac{1}{q}}d\mu, \tag{3.14}$$

$$\Omega_2(\lambda)=\int_{\lambda}^{1}\left|\mu-\frac{4-3\lambda}{6(1-\lambda)}\right|(1-\mu)^{\frac{1}{q}}d\mu, \tag{3.15}$$

$$\Omega_3(\lambda)=\int_0^{\lambda}\left|\mu-\frac{3\lambda-1}{6\lambda}\right|\mu^{\frac{1}{q}}d\mu, \tag{3.16}$$

$$\Omega_4(\lambda)=\int_{\lambda}^{1}\left|\mu-\frac{4-3\lambda}{6(1-\lambda)}\right|\mu^{\frac{1}{q}}d\mu, \tag{3.17}$$

$$\Omega_5(\lambda)=\int_0^{\lambda}\left|\mu-\frac{3\lambda-1}{6\lambda}\right|\mu^{\frac{1}{2q}}(1-\mu)^{\frac{1}{2q}}d\mu \tag{3.18}$$

and

$$\Omega_6(\lambda)=\int_{\lambda}^{1}\left|\mu-\frac{4-3\lambda}{6(1-\lambda)}\right|\mu^{\frac{1}{2q}}(1-\mu)^{\frac{1}{2q}}d\mu. \tag{3.19}$$

The proof is completed. □

Theorem 3.10. *Let $\mathfrak{h}$ be as in Lemma 2.2. If $|\mathfrak{h}'|^q$ is convex, where $\frac{1}{2}<q\leq 1$, then we have*

$$\left|\frac{3\lambda-1}{6\lambda}\mathfrak{h}(\iota_1)+\frac{1}{6\lambda(1-\lambda)}\mathfrak{h}((1-\lambda)\iota_1+\lambda\iota_2)+\frac{2-3\lambda}{6(1-\lambda)}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right|$$

$$\leq(\iota_2-\iota_1)\Big((\Omega_1(\lambda)+\Omega_2(\lambda))\left|\mathfrak{h}'(\iota_1)\right|+(\Omega_3(\lambda)+\Omega_4(\lambda))\left|\mathfrak{h}'(\iota_2)\right|$$

$$+\left(2^{\frac{1}{q}}-2\right)\left(\Omega_5(\lambda)+\Omega_6(\lambda)\right)\left(\left|\mathfrak{h}'(\iota_1)\right|\left|\mathfrak{h}'(\iota_2)\right|\right)^{\frac{1}{2}}\Big),$$

where $\Omega_i(\lambda)$ *for* $i=1$ *to* 6 *are defined by* (3.14)-(3.19)*, respectively.*

Proof. Since, $|\mathfrak{h}'|^q$ is convex with $\frac{1}{2}<q\leq 1$, then we have from Lemma 2.1

$$\begin{aligned}&\left|\mathfrak{h}'\left((1-\mu)\iota_1+\mu\iota_2\right)\right| \\ &\leq (1-\mu)^{\frac{1}{q}}\left|\mathfrak{h}'(\iota_1)\right|+\mu^{\frac{1}{q}}\left|\mathfrak{h}'(\iota_2)\right|+\left(2^{\frac{1}{q}}-2\right)\mu^{\frac{1}{2q}}(1-\mu)^{\frac{1}{2q}}\left(\left|\mathfrak{h}'(\iota_1)\right|\left|\mathfrak{h}'(\iota_2)\right|\right)^{\frac{1}{2}}.\end{aligned} \tag{3.20}$$

Using (3.20) into (3.13), we obtain

$$\begin{aligned}&\left|\frac{3\lambda-1}{6\lambda}\mathfrak{h}(\iota_1)+\frac{1}{6\lambda(1-\lambda)}\mathfrak{h}\left((1-\lambda)\iota_1+\lambda\iota_2\right)+\frac{2-3\lambda}{6(1-\lambda)}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right| \\
&\leq(\iota_2-\iota_1)\Bigg(\int_0^{\lambda}\left|\left(\mu-\tfrac{3\lambda-1}{6\lambda}\right)\right|\left((1-\mu)^{\frac{1}{q}}\left|\mathfrak{h}'(\iota_1)\right|+\mu^{\frac{1}{q}}\left|\mathfrak{h}'(\iota_2)\right|+\left(2^{\frac{1}{q}}-2\right)\mu^{\frac{1}{2q}}(1-\mu)^{\frac{1}{2q}}\left(\left|\mathfrak{h}'(\iota_1)\right|\left|\mathfrak{h}'(\iota_2)\right|\right)^{\frac{1}{2}}\right)d\mu \\
&\quad+\int_{\lambda}^{1}\left|\left(\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right)\right|\left((1-\mu)^{\frac{1}{q}}\left|\mathfrak{h}'(\iota_1)\right|+\mu^{\frac{1}{q}}\left|\mathfrak{h}'(\iota_2)\right|+\left(2^{\frac{1}{q}}-2\right)\mu^{\frac{1}{2q}}(1-\mu)^{\frac{1}{2q}}\left(\left|\mathfrak{h}'(\iota_1)\right|\left|\mathfrak{h}'(\iota_2)\right|\right)^{\frac{1}{2}}\right)d\mu\Bigg) \\
&=(\iota_2-\iota_1)\Bigg(\left|\mathfrak{h}'(\iota_1)\right|\int_0^{\lambda}\left|\left(\mu-\tfrac{3\lambda-1}{6\lambda}\right)\right|(1-\mu)^{\frac{1}{q}}d\mu+\left|\mathfrak{h}'(\iota_2)\right|\int_0^{\lambda}\left|\left(\mu-\tfrac{3\lambda-1}{6\lambda}\right)\right|\mu^{\frac{1}{q}}d\mu \\
&\quad+\left(2^{\frac{1}{q}}-2\right)\left(\left|\mathfrak{h}'(\iota_1)\right|\left|\mathfrak{h}'(\iota_2)\right|\right)^{\frac{1}{2}}\int_{\lambda}^{1}\left|\left(\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right)\right|\mu^{\frac{1}{2q}}(1-\mu)^{\frac{1}{2q}}d\mu \\
&\quad+\left|\mathfrak{h}'(\iota_1)\right|\int_{\lambda}^{1}\left|\left(\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right)\right|(1-\mu)^{\frac{1}{q}}d\mu+\left|\mathfrak{h}'(\iota_2)\right|\int_{\lambda}^{1}\left|\left(\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right)\right|\mu^{\frac{1}{q}}d\mu \\
&\quad+\left(2^{\frac{1}{q}}-2\right)\left(\left|\mathfrak{h}'(\iota_1)\right|\left|\mathfrak{h}'(\iota_2)\right|\right)^{\frac{1}{2}}\int_{\lambda}^{1}\left|\left(\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right)\right|\mu^{\frac{1}{2q}}(1-\mu)^{\frac{1}{2q}}d\mu\Bigg) \\
&=(\iota_2-\iota_1)\Big(\left(\Omega_1(\lambda)+\Omega_2(\lambda)\right)\left|\mathfrak{h}'(\iota_1)\right|+\left(\Omega_3(\lambda)+\Omega_4(\lambda)\right)\left|\mathfrak{h}'(\iota_2)\right| \\
&\quad+\left(2^{\frac{1}{q}}-2\right)\left(\Omega_5(\lambda)+\Omega_6(\lambda)\right)\left(\left|\mathfrak{h}'(\iota_1)\right|\left|\mathfrak{h}'(\iota_2)\right|\right)^{\frac{1}{2}}\Big),\end{aligned}$$

where we have used (3.14)-(3.19). □

Theorem 3.11. *Let* $\mathfrak{h}$ *be as in Lemma 2.2. If* $|\mathfrak{h}'|^q$ *is convex where* $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$*, then we have*

$$\begin{aligned}&\left|\frac{3\lambda-1}{6\lambda}\mathfrak{h}(\iota_1)+\frac{1}{6\lambda(1-\lambda)}\mathfrak{h}\left((1-\lambda)\iota_1+\lambda\iota_2\right)+\frac{2-3\lambda}{6(1-\lambda)}\mathfrak{h}(\iota_2)-\tfrac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right| \\ &\leq(\iota_2-\iota_1)\left(\frac{\Xi_1(\lambda,p)+\Xi_2(\lambda,p)}{p}+\frac{\left|\mathfrak{h}'(\iota_1)\right|^q+\left|\mathfrak{h}'(\iota_2)\right|^q}{2q}\right),\end{aligned}$$

where $\Xi_1(\lambda,p)$ *and* $\Xi_2(\lambda,p)$ *are defined by* (3.5) *and* (3.6)*, respectively.*

Proof. From Lemma 2.2, Young's inequality along with the convexity of $|\mathfrak{h}'|^q$, it yields

$$\left|\frac{3\lambda-1}{6\lambda}\mathfrak{h}(\iota_1)+\frac{1}{6\lambda(1-\lambda)}\mathfrak{h}((1-\lambda)\iota_1+\lambda\iota_2)+\frac{2-3\lambda}{6(1-\lambda)}\mathfrak{h}(\iota_2)-\frac{1}{\iota_2-\iota_1}\int_{\iota_1}^{\iota_2}\mathfrak{h}(u)\,du\right|$$

$$\leq(\iota_2-\iota_1)\left(\int_0^\lambda\left|\left(\mu-\tfrac{3\lambda-1}{6\lambda}\right)\right|\left|\mathfrak{h}'((1-\mu)\iota_1+\mu\iota_2)\right|d\mu+\int_\lambda^1\left|\left(\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right)\right|\left|\mathfrak{h}'((1-\mu)\iota_1+\mu\iota_2)\right|d\mu\right)$$

$$\leq(\iota_2-\iota_1)\left(\frac{1}{p}\int_0^\lambda\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|^p d\mu+\frac{1}{q}\int_0^\lambda\left|\mathfrak{h}'((1-\mu)\iota_1+\mu\iota_2)\right|^q d\mu\right.$$

$$\left.+\frac{1}{p}\int_\lambda^1\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|^p d\mu+\frac{1}{q}\int_\lambda^1\left|\mathfrak{h}'((1-\mu)\iota_1+\mu\iota_2)\right|^q d\mu\right)$$

$$\leq(\iota_2-\iota_1)\left(\frac{1}{p}\int_0^\lambda\left|\mu-\tfrac{3\lambda-1}{6\lambda}\right|^p d\mu+\frac{1}{q}\int_0^\lambda\left((1-\mu)\left|\mathfrak{h}'(\iota_1)\right|^q+\mu\left|\mathfrak{h}'(\iota_2)\right|^q\right)d\mu\right.$$

$$\left.+\frac{1}{p}\int_\lambda^1\left|\mu-\tfrac{4-3\lambda}{6(1-\lambda)}\right|^p d\mu+\frac{1}{q}\int_\lambda^1\left((1-\mu)\left|\mathfrak{h}'(\iota_1)\right|^q+\mu\left|\mathfrak{h}'(\iota_2)\right|^q\right)d\mu\right)$$

$$=(\iota_2-\iota_1)\left(\tfrac{\Xi_1(\lambda,p)}{p}+\tfrac{\left(1-(1-\lambda)^2\right)|\mathfrak{h}'(\iota_1)|^q+\lambda^2|\mathfrak{h}'(\iota_2)|^q}{2q}+\tfrac{\Xi_2(\lambda,p)}{p}+\tfrac{(1-\lambda)^2|\mathfrak{h}'(\iota_1)|^q+\left(1-\lambda^2\right)|\mathfrak{h}'(\iota_2)|^q}{2q}\right)$$

$$=(\iota_2-\iota_1)\left(\frac{\Xi_1(\lambda,p)+\Xi_2(\lambda,p)}{p}+\frac{|\mathfrak{h}'(\iota_1)|^q+|\mathfrak{h}'(\iota_2)|^q}{2q}\right),$$

which is the desired result. □

4. Illustrative examples

To demonstrate the effectiveness and accuracy of the proposed inequalities, this section presents a series of illustrative examples, supported by graphical representations.

Example 4.1. Let $\mathfrak{h}$ be a function defined on $[0,1]$ by $\mathfrak{h}(u)=u^3$. This function fulfills the condition of Theorem 3.1 since its derivative $\mathfrak{h}'=3u^2$ satisfies the condition $|\mathfrak{h}'|$ is convex on $[0,1]$. From Theorem 3.1, we have

$$\left|\frac{1-2\lambda}{12}\right|\leq 3\left(\Lambda_3(\lambda)+\Lambda_4(\lambda)\right), \tag{4.1}$$

where Λ_3 and Λ_4 are defined as in (3.2) and (3.3), respectively.

The left-hand side and the right-hand side of inequality (4.1) are plotted in Figure 1.

Example 4.2. Let $\mathfrak{h}$ be a function defined on the interval $[1,2]$ by $\mathfrak{h}(u)=\frac{1}{u}$. This function satisfies the assumptions of Theorem 3.1, as its derivative, given by $\mathfrak{h}'(u)=-\frac{1}{u^2}$, has an absolute value that is convex on $[1,2]$. Applying Theorem 3.1, we obtain the inequality

$$\left|\frac{3\lambda-1}{6\lambda}+\frac{1}{6\lambda(1-\lambda^2)}+\frac{2-3\lambda}{12(1-\lambda)}-\ln(2)\right|\leq\frac{4\left(\Lambda_3(\lambda)+\Lambda_4(\lambda)\right)+\left(\Lambda_3(\lambda)+\Lambda_4(\lambda)\right)}{4}, \tag{4.2}$$

where the functions Λ_1, Λ_2, Λ_3, and Λ_4 are defined as in equations (2.4)-(3.3), respectively.

Figure 2 provides a graphical representation of both sides of inequality (4.2).

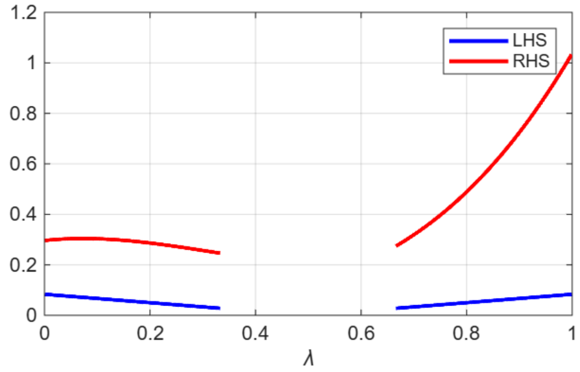

Figure 1: Graphical illustration of Example 4.1

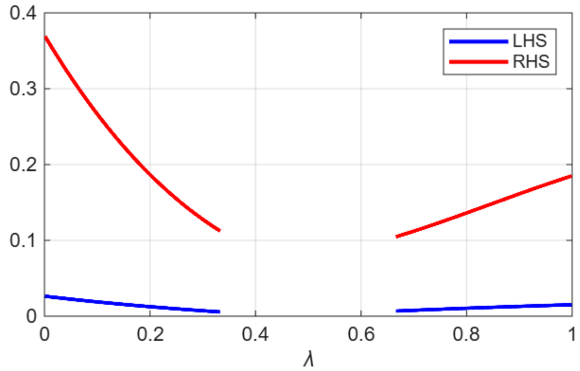

Figure 2: Graphical illustration of Example 4.2

5. Applications

In this section, we apply the derived inequalities to establish bounds for special means, highlighting their relevance and applicability in mathematical analysis.

The means for arbitrary real numbers $\imath_1, \imath_2$ will be examined.

The formula for the weighted arithmetic mean is given by $\mathscr{A}(n,m,\imath_1,\imath_2) = \frac{n\imath_1+m\imath_2}{n+m}$.

The p-Logarithmic mean is expressed as: $\mathscr{L}_p(\imath_1,\imath_2) = \left(\frac{\imath_2-\imath_1}{(p+1)(\imath_2-\imath_1)}\right)^{\frac{1}{p}}$, $\imath_1,\imath_2 > 0$, $\imath_1 \neq \imath_2$ and $p \in \mathbb{R}\backslash\{-1,0\}$.

Proposition 5.1. *For real numbers $\imath_1,\imath_2,n$ such that $0<\imath_1<\imath_2$ and $n \geq 2$, we have*

$$\left|\frac{\imath_2^n-\imath_1^n}{6} + \mathscr{A}\left(16,2,\mathscr{A}^n(3,1,\imath_1,\imath_2),\imath_2\right) - \mathscr{L}_n^n(\imath_1,\imath_2)\right|$$

$$\leq \frac{n(\imath_2-\imath_1)}{303750}\left(23261\imath_1^{n-1} + 10440\imath_2^{n-1}\right).$$

Proof. The desired result can be readily obtained by applying Theorem 3.1 with $\lambda = \frac{1}{4}$ to the function $\mathfrak{h}(u) = u^n$. □

Proposition 5.2. *For real numbers $\imath_1,\imath_2,n$ such that $0<\imath_1<\imath_2$ and $n \geq 2$, we have*

$$\left|\frac{\imath_2^n-\imath_1^n}{6} + \mathscr{A}\left(16,2,\mathscr{A}^n(3,1,\imath_1,\imath_2),\imath_2\right) - \mathscr{L}_n^n(\imath_1,\imath_2)\right|$$

$$\leq \frac{n(\imath_2-\imath_1)}{144}\left(\frac{30^3+34^3+20^3-12^3}{108}\right)^{\frac{1}{2}}\left(\imath_1^{2n-2}+\imath_2^{2n-2}\right)^{\frac{1}{2}}.$$

Proof. The assertion follows directly from an application of Theorem 3.7, with $p=q=2$ and $\lambda = \frac{1}{4}$ to the function $\mathfrak{h}(u) = u^n$. □

6. Conclusion

In summary, this work introduces a new integral identity and develops associated Radau-type inequalities under convexity assumptions on the first derivative. The proposed approach yields several interesting results that are consistent with known inequalities in the literature, while offering new perspectives and refinements. Furthermore, we have illustrated the applicability of these inequalities through examples involving means. We believe that this contribution opens up promising directions for future investigations in the theory of integral inequalities and their practical uses.

Use of AI Tools

AI tools were not employed in generating, analyzing, or interpreting the results.

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