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Research Article

Some Integral Inequalities via Caputo and Riemann-Liouville Fractional Integral Operators for m-convex Functions

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Abstract

The present study is comprised of two sections. Firstly, this study aims is to obtain some inequalities on Caputo fractional derivatives using elementary inequalities. Secondly, several novel inequalities are established, including Caputo fractional derivatives for m-convex functions. In this paper, upper bounds of the Caputo type for Lemma 1.8 [28] and Lemma 1.9 [29] have been obtained.

Keywords: Caputo fractional derivative, m-convex function, Hölder inequality, power-mean inequality

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1. Introduction

In a variety of disciplines within the field of mathematical analysis, including but not limited to differential equation theory, approximation theory and, most notably, fractional calculus or the calculus of non-integer order, inequalities of all forms play a pivotal role. The theory of fractional calculus has attracted a considerable degree of interest due to its numerous applications in the applied sciences. This theory studies and applies derivatives and integrals of arbitrary orders. Fractional differential equations have recently been used extensively in a variety of applied disciplines to describe actual systems. The existence, uniqueness, and stability of solutions to a system of fractional differential equations have been examined using fractional inequalities, commonly referred to as the inequalities involving derivatives and integrals of arbitrary orders. Fractional inequalities are frequently used to determine the upper and lower limits of solutions to a system of fractional differential equations. Fractional inequalities are also used in probability, numerical quadrature, and many other related fields. Over the years, a large number of authors have created various extensions of the various classical inequalities to fractional calculus in the literature. In this regard, there are several definitions for fractional integral operators, such as Riemann-Liouville [1], k-Riemann-Liouville [2], Hadamard [3], conformable [4], Caputo [5] and Caputo-Fabrizio fractional integrals [6]. Numerous research on this subject are available in the appropriate resources for those who are curious about the trends discussed above. References [7, 12] provide a sample of these investigations.

Today, the concept of FC dates back to Leibniz. Leibniz discussed the concept of FC with his contemporaries in 1695. Euler noticed in 1738 what a problem non-integer order derivatives FC pose. Abel in 1826 and Liouville in 1832 gave versions of the non-integer order derivative.

Let us present the necessary definitions and preliminary information that we will use in this study.

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Definition 1.1. A function $f : \mathscr{I} \to \mathbb{R}$ is said to be convex function on $\mathscr{I}$ if the inequality

$$f(\gamma\varkappa + (1-\gamma)\vartheta) \leq \gamma f(\varkappa) + (1-\gamma) f(\vartheta)$$

holds for all $\varkappa, \vartheta \in \mathscr{I}$ and $\gamma \in [0,1]$.

Definition 1.2. ([13]) The function $f : [0,\mu] \to \mathbb{R}$, $\mu > 0$, is said to be m-convex where $m \in [0,1]$, if we have

$$f(\gamma\varkappa + m(1-\gamma)\vartheta) \leq \gamma f(\varkappa) + m(1-\gamma) f(\vartheta)$$

for all $\varkappa, \vartheta \in [0,\mu]$ and $\gamma \in [0,1]$. We say that f is m-concave if $(-f)$ is m-convex. When $m = 1$, then this reduces the definition of classical convex function.

For many papers connected with m-convex see for Hermite-Hadamard type [14, 15], Jensen type [16], fundamental definition of m-convex [17], definition of co-ordinated m-convex [18], and different form of m-convexity [19, 20] and the references therein.

We will need the modified forms of the m-convex function as follows:

The inequalities obtained below will be used throughout the paper

$m-$convexity of f :

$$f(\gamma\varkappa + (1-\gamma)\vartheta) = f\left(\gamma\varkappa + m(1-\gamma)\frac{\vartheta}{m}\right) \leq \gamma f(\varkappa) + m(1-\gamma) f\left(\frac{\vartheta}{m}\right),$$

m-convexity of $\left|f^{(\eta+1)}\right|$:

$$\left|f^{(\eta+1)}(\gamma\varkappa + (1-\gamma)\vartheta)\right| = \left|f^{(\eta+1)}\left(\gamma\varkappa + m(1-\gamma)\frac{\vartheta}{m}\right)\right| \leq \gamma\left|f^{(\eta+1)}(\varkappa)\right| + m(1-\gamma)\left|f^{(\eta+1)}\left(\frac{\vartheta}{m}\right)\right|,$$

m-convexity of $\left|f^{(\eta+1)}\right|^q$:

$$\left|f^{(\eta+1)}(\gamma\varkappa + (1-\gamma)\vartheta)\right|^q = \left|f^{(\eta+1)}\left(\gamma\varkappa + m(1-\gamma)\frac{\vartheta}{m}\right)\right|^q \leq \gamma\left|f^{(\eta+1)}(\varkappa)\right|^q + m(1-\gamma)\left|f^{(\eta+1)}\left(\frac{\vartheta}{m}\right)\right|^q$$

where $m \in (0,1]$.

M. Caputo, in 1967, made the most significant contribution to the field of fractional calculus. A salient disadvantage of the Riemann-Liouville definition of fractional derivative pertains to the unconventional set of initial conditions that it necessitates. Caputo's seminal work in the field involved a reformulation of the traditional definition of the Riemann-Liouville fractional derivative, with the innovative application of classical initial conditions [21]. The following definition of fractional calculus theory is recalled, as it is one that is used extensively in the present paper.

Definition 1.3. ([22]) Let $\alpha > 0$ and $\alpha \notin \{1,2,3,...\}$, $\eta = [\alpha] + 1$, $f \in C^{\eta}[\sigma,\mu]$, the space of functions whose η-th derivatives absolutely continuous. The left-sided and right-sided Caputo fractional derivatives of order α are defined as follows:

$$\left({}^{C}\mathscr{D}^{\alpha}_{\sigma+} f\right)(\varkappa) = \frac{1}{\Gamma(\eta-\alpha)} \int_{\sigma}^{\varkappa} \frac{f^{(\eta)}(\gamma)}{(\varkappa-\gamma)^{\alpha-\eta+1}} d\gamma, \;\; \varkappa > \sigma$$

and

$$\left({}^{C}\mathscr{D}^{\alpha}_{\mu-} f\right)(\varkappa) = \frac{(-1)^{\eta}}{\Gamma(\eta-\alpha)} \int_{\varkappa}^{\mu} \frac{f^{(\eta)}(\gamma)}{(\gamma-\varkappa)^{\alpha-\eta+1}} d\gamma, \;\; \varkappa < \mu.$$

If $\eta = 1$ and $\alpha = 0$, we have $\left({}^{C}D^{0}_{\sigma+} f\right)(\varkappa) = \left({}^{C}D^{0}_{\mu-} f\right)(\varkappa) = f(\varkappa)$. Some research work related to the Caputo fractional operators can be found in [23, 27] and the references therein.

An essential tool for the study of L^p spaces is Hölder's inequality, a basic inequality between integrals. By employing various convex functions and this inequality, several new developments and improvements have been made to the theory of inequalities. The following is the Hölder inequality:

Theorem 1.4. *(Hölder Inequality) Let be $p,q>1$ and $p^{-1}+q^{-1}=1$. If f and g real valued functions on $[\sigma,\mu]$ such that $|f|^p$ and $|f|^q$ are integrable on $[\sigma,\mu]$, then*

$$\int_{\sigma}^{\mu}|f(\varkappa)g(\varkappa)|\,d\varkappa\leq\left(\int_{\sigma}^{\mu}|f(\varkappa)|^{p}\,d\varkappa\right)^{\frac{1}{p}}\left(\int_{\sigma}^{\mu}|g(\varkappa)|^{q}\,d\varkappa\right)^{\frac{1}{q}}.$$

Furthermore, a variant of the Hölder integral inequality, the power-mean integral inequality, is important in many areas of mathematical analysis, especially convex analysis. The power-mean inequality is presented as follows:

Theorem 1.5. *(Power-mean Inequality for Integrals) Let $q\geq 1$. If f and g are real functions defined on $[\sigma,\mu]$ such that $|f|$ and $|g|^q$ are integrable functions on $[\sigma,\mu]$, then:*

$$\int_{\sigma}^{\mu}|f(\varkappa)g(\varkappa)|\,d\varkappa\leq\left(\int_{\sigma}^{\mu}|f(\varkappa)|\,d\varkappa\right)^{1-\frac{1}{q}}\left(\int_{\sigma}^{\mu}|f(\varkappa)g(\varkappa)|^{q}\,d\varkappa\right)^{\frac{1}{q}}.$$

Definition 1.6. (Beta Function) The Beta function denoted by $\beta(\sigma,\mu)$ is defined as

$$\beta(\sigma,\mu)=\int_{0}^{1}\gamma^{\alpha-1}(1-\gamma)^{\mu-1}\,d\gamma,\ \sigma,\mu>0.$$

Corollary 1.7. *Beta function provides the following properties:*

1. $\beta(\sigma,\mu)=\beta(\mu,\sigma)$
2. $\beta(\sigma+1,\mu)=\frac{\sigma}{\sigma+\mu}\beta(\sigma,\mu)$.

In [28], Farid *et al.* established the following identity for Caputo fractional operators.

Lemma 1.8. *Let $f:[\sigma,\mu]\to\mathbb{R}$, be a differentiable mapping on (σ,μ) with $0\leq\sigma<\mu$. If $f^{(\eta+1)}\in L[\sigma,\mu]$, then the following equality for fractional integrals holds:*

$$\begin{aligned}&\frac{f^{(\eta)}(\sigma)+f^{(\eta)}(\mu)}{2}-\frac{\Gamma(\eta-\alpha+1)}{2(\mu-\sigma)^{\eta-\alpha}}\left[({}^{C}\mathscr{D}_{\sigma^{+}}^{\alpha}f)(\mu)+(-1)^{\eta}({}^{C}\mathscr{D}_{\mu^{-}}^{\alpha}f)(\sigma)\right]\\ =\ &\frac{\mu-\sigma}{2}\int_{0}^{1}\left[(1-\gamma)^{\eta-\alpha}-\gamma^{\eta-\alpha}\right]f^{(\eta+1)}(\gamma\sigma+(1-\gamma)\mu)\,d\gamma.\end{aligned}$$

In [29], authors obtained a new identity to utilize different types of convex functions for left-sided Caputo derivatives in Lemma 2 as follows:

Lemma 1.9. *Let $f:\mathscr{I}\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I ,where $\sigma,\mu\in\mathscr{I}$ with $\gamma\in[0,1]$. If $f^{(\eta+1)}\in L[\sigma,\mu]$, then for all $\sigma\leq\varkappa<\vartheta\leq\mu$ and $\alpha>0$ we have*

$$\frac{1}{\vartheta-\varkappa}f^{(\eta)}(\vartheta)-\frac{(-1)^{\eta}\Gamma(\eta-\alpha+1)}{(\vartheta-\varkappa)^{\eta-\alpha+1}}({}^{C}\mathscr{D}_{\vartheta^{-}}^{\alpha}f)(\varkappa)=\int_{0}^{1}(1-\gamma)^{\eta-\alpha}f^{(\eta+1)}(\gamma\varkappa+(1-\gamma)\vartheta)\,d\gamma.$$

In this study, we establish certain inequalities for both left- and right-sided Caputo derivatives in different ways. Also, this paper aims at establishing new upper boundaries. We employed some classical inequalities to do this.

2. Further results

Theorem 2.1. *Let $f:\mathscr{I}\subset[0,\infty)\to\mathbb{R}$ be a differentiable function on $\mathscr{I}$ such that $f\in C^{\eta}[\sigma,\mu]$ where $\sigma,\mu\in\mathscr{I}$ with $0<\sigma<\gamma<\varkappa\leq\mu$. Then we obtain*

$$\begin{aligned}\int_{\sigma}^{\mu}f^{(\eta)}(\gamma)\,d\gamma\ \leq\ &\frac{\Gamma(\eta-\alpha)}{2}\left[({}^{C}\mathscr{D}_{\sigma^{+}}^{\alpha}f)(\varkappa)+(-1)^{\eta}({}^{C}\mathscr{D}_{\mu^{-}}^{\alpha}f)(\varkappa)\right]\\ &+\frac{\Gamma(\alpha-\eta+2)}{2}\left[({}^{C}\mathscr{D}_{\sigma^{+}}^{\alpha}f)(\varkappa)+(-1)^{\eta}({}^{C}\mathscr{D}_{\mu^{-}}^{\alpha}f)(\varkappa)\right]\end{aligned}\tag{2.1}$$

where $\alpha>0$, $\alpha\notin\{1,2,3,...\}$ and $\eta=[\alpha]+1$, $f^{(\eta)}>0$.

Proof. First of all, since $(\varkappa-\gamma)>0$, we can write the following elementary inequality

$$(\varkappa-\gamma)^{\eta-\alpha-1}+\frac{1}{(\varkappa-\gamma)^{\eta-\alpha-1}}=(\varkappa-\gamma)^{\eta-\alpha-1}+(\varkappa-\gamma)^{\alpha-\eta+1}\geq 2.$$

Now, if we multiply both sides of the final inequality by $f^{(\eta)}>0$ and then integrate it over $[\sigma,\mu]$, we have

$$\begin{aligned}
2\int_{\sigma}^{\mu} f^{(\eta)}(\gamma)\,d\gamma &\leq \int_{\sigma}^{\mu}(\varkappa-\gamma)^{\eta-\alpha-1} f^{(\eta)}(\gamma)\,d\gamma+\int_{\sigma}^{\mu}(\varkappa-\gamma)^{\alpha-\eta+1} f^{(\eta)}(\gamma)\,d\gamma\\
&= \int_{\sigma}^{\varkappa}(\varkappa-\gamma)^{\eta-\alpha-1} f^{(\eta)}(\gamma)\,d\gamma+\int_{\varkappa}^{\mu}(\varkappa-\gamma)^{\alpha-\eta-1} f^{(\eta)}(\gamma)\,d\gamma\\
&\quad+\int_{\sigma}^{\varkappa}(\varkappa-\gamma)^{\alpha-\eta+1} f^{(\eta)}(\gamma)\,d\gamma+\int_{\varkappa}^{\mu}(\varkappa-\gamma)^{\alpha-\eta+1} f^{(\eta)}(\gamma)\,d\gamma\\
&= \Gamma(\eta-\alpha)\left({}^{C}\mathscr{D}_{\sigma^+}^{\alpha}f\right)(\varkappa)+(-1)^{\eta}\,\Gamma(\eta-\alpha)\left({}^{C}\mathscr{D}_{\mu^-}^{\alpha}f\right)(\varkappa)\\
&\quad+\Gamma(\alpha-\eta+2)\left({}^{C}\mathscr{D}_{\sigma^+}^{\alpha}f\right)(\varkappa)+(-1)^{\alpha}\,\Gamma(\alpha-\eta+2)\left({}^{C}\mathscr{D}_{\mu^-}^{\alpha}f\right)(\varkappa).
\end{aligned}$$

Taking into account Definition 1.3, we obtain inequality (2.1). □

Theorem 2.2. *Let* $\alpha>0$, $\alpha\notin\{1,2,3,...\}$ *and* $\eta=[\alpha]+1$, $f^{(\eta)}>0$. *Let* $f:\mathscr{I}\subset[0,\infty)\to\mathbb{R}$ *be a differentiable function on I such that* $f\in C^{\eta}[\sigma,\mu]$ *where* $\sigma,\mu\in\mathscr{I}$ *with* $0<\gamma\leq\sigma\leq\varkappa\leq\mu$. *Then the following inequality holds:*

$$\int_{\sigma}^{\mu}\sqrt{|(\varkappa-\gamma)|^{2(\eta-\alpha)}}\,d\gamma\leq\frac{\Gamma(\eta-\alpha+1)}{2}\left[\left({}^{C}\mathscr{D}_{\sigma^+}^{\alpha}f\right)(\varkappa)+(-1)^{\eta}\left({}^{C}\mathscr{D}_{\mu^-}^{\alpha}f\right)(\varkappa)\right]. \tag{2.2}$$

Proof. According to relation between the Geometric and Arithmetic means, we can write the basic inequality as follows:

$$\begin{aligned}
\sqrt{|(\varkappa-\gamma)|^{2(\eta-\alpha)}} &= \sqrt{|(\varkappa-\gamma)|^{(\eta-\alpha)}\,|(\gamma-\varkappa)|^{(\eta-\alpha)}}\\
&\leq \frac{1}{2}\left[|(\varkappa-\gamma)|^{(\eta-\alpha)}+|(\gamma-\varkappa)|^{(\eta-\alpha)}\right]\\
&\leq \frac{1}{2}\left[|(\varkappa-\gamma)|^{(\eta-\alpha)}+|(\gamma-\varkappa)|^{(\eta-\alpha)}\right]f^{(\eta)}(\gamma),\quad\left(f^{(\eta)}>0\right)\\
&= \frac{1}{2}\left[|(\varkappa-\gamma)|^{(\eta-\alpha)}f^{(\eta)}(\gamma)+|(\gamma-\varkappa)|^{(\eta-\alpha)}f^{(\eta)}(\gamma)\right].
\end{aligned}$$

Now, if we integrate both sides of the first and last terms over $[\sigma,\mu]$, we obtain

$$\begin{aligned}
\int_{\sigma}^{\mu}\sqrt{|(\varkappa-\gamma)|^{2(\eta-\alpha)}}\,d\gamma &\leq \frac{1}{2}\left[\int_{\sigma}^{\mu}|(\varkappa-\gamma)|^{(\eta-\alpha)}f^{(\eta)}(\gamma)\,d\gamma+\int_{\sigma}^{\mu}|(\gamma-\varkappa)|^{(\eta-\alpha)}f^{(\eta)}(\gamma)\,d\gamma\right]\\
&= \frac{1}{2}\left[\int_{\sigma}^{\varkappa}|(\varkappa-\gamma)|^{(\eta-\alpha)}f^{(\eta)}(\gamma)\,d\gamma+\int_{\varkappa}^{\mu}|(\varkappa-\gamma)|^{(\eta-\alpha)}f^{(\eta)}(\gamma)\,d\gamma\right]\\
&\quad+\frac{1}{2}\left[\int_{\sigma}^{\varkappa}|(\gamma-\varkappa)|^{(\eta-\alpha)}f^{(\eta)}(\gamma)\,d\gamma+\int_{\varkappa}^{\mu}|(\gamma-\varkappa)|^{(\eta-\alpha)}f^{(\eta)}(\gamma)\,d\gamma\right]\\
&= \frac{1}{2}\left[\int_{\sigma}^{\varkappa}(\varkappa-\gamma)^{\eta-\alpha}f^{(\eta)}(\gamma)\,d\gamma-\int_{\mu}^{\varkappa}(\varkappa-\gamma)^{\eta-\alpha}f^{(\eta)}(\gamma)\,d\gamma\right]\\
&\quad+\frac{1}{2}\left[(\gamma-\varkappa)^{\eta-\alpha}f^{(\eta)}(\gamma)\,d\gamma-\int_{\mu}^{\varkappa}|(\gamma-\varkappa)|^{(\eta-\alpha)}f^{(\eta)}(\gamma)\,d\gamma\right]\\
&= \frac{\Gamma(\eta-\alpha+1)}{2}\left[\left({}^{C}\mathscr{D}_{\sigma^+}^{\alpha}f\right)(\varkappa)+(-1)^{\eta}\left({}^{C}\mathscr{D}_{\mu^-}^{\alpha}f\right)(\varkappa)\right].
\end{aligned}$$

This completes the proof of inequality (2.2). □

3. New results for m-convex functions

This section deals with deriving new inequalities for differentiable m-convex functions that involve Caputo fractional operators. Then, taking these inequalities into account and with the help of some fundamental integral inequalities, such as Hölder's inequality, power-mean inequality, Lemma 1.8, and Lemma 1.9, several refinements are presented.

Theorem 3.1. *Let $f:[\sigma,\mu]\to\mathbb{R}$, be a differentiable function on $\mathscr{I}$ such that $f^{(\eta+1)}\in L[\sigma,\mu]$. If $\left|f^{(\eta+1)}\right|$ is m-convex function for $\gamma\in[0,1]$, then for all $\alpha>0$, $\alpha\notin\{1,2,3,...\}$ and $\eta=[\alpha]+1$, $m\in(0,1]$, we have*

$$\left|\frac{f^{(\eta)}(\sigma)+f^{\eta}(\mu)}{2}-\frac{\Gamma(\alpha-\eta+1)}{2(\mu-\sigma)^{\eta-\alpha}}\left[({}^{C}\mathscr{D}^{\alpha}_{\sigma^{+}}f)(\mu)+(-1)^{\eta}({}^{C}\mathscr{D}^{\alpha}_{\mu^{-}}f)(\sigma)\right]\right| \tag{3.1}$$
$$\leq \frac{\mu-\sigma}{4(\eta-\alpha+1)}\left(\left|f^{(\eta+1)}(\sigma)\right|+m\left|f^{(\eta+1)}\left(\frac{\mu}{m}\right)\right|\right).$$

Proof. We know from our elementary knowledge that for $\alpha\in[0,1]$ and $\forall\gamma_1,\gamma_2\in[0,1]$, $\left|\gamma_1^{\eta-\alpha}-\gamma_2^{\eta-\alpha}\right|\leq|\gamma_1-\gamma_2|^{\eta-\alpha}$. Let be

$$\Sigma(f,\Gamma,\sigma,\mu)=\frac{f^{(\eta)}(\sigma)+f^{(\eta)}(\mu)}{2}-\frac{\Gamma(\eta-\alpha+1)}{2(\mu-\sigma)^{\eta-\alpha}}\left[({}^{C}\mathscr{D}^{\alpha}_{\sigma^{+}}f)(\mu)+({}^{C}\mathscr{D}^{\alpha}_{\mu^{-}}f)(\sigma)\right].$$

In Lemma 1.8, using the properties of the modulus as well as the fact that $\left|f^{(\eta+1)}\right|$ is m-convex on $[\sigma,\mu]$, we can write the relation below:

$$\begin{aligned}
|\Sigma(f,\Gamma,\sigma,\mu)| &\leq \frac{\mu-\sigma}{2}\int_0^1\left|(1-\gamma)^{\eta-\alpha}-\gamma^{\eta-\alpha}\right|\left|f^{(\eta+1)}(\gamma\sigma+(1-\gamma)\mu)\right|d\gamma\\
&\leq \frac{\mu-\sigma}{2}\int_0^1|1-2\gamma|^{\eta-\alpha}\left|f^{(\eta+1)}(\gamma\sigma+(1-\gamma)\mu)\right|d\gamma\\
&= \frac{\mu-\sigma}{2}\int_0^1|1-2\gamma|^{\eta-\alpha}\left|f^{(\eta+1)}\left(\gamma\sigma+m(1-\gamma)\frac{\mu}{m}\right)\right|d\gamma\\
&\leq \frac{\mu-\sigma}{2}\int_0^1|1-2\gamma|^{\eta-\alpha}\left[\gamma\left|f^{(\eta+1)}(\sigma)\right|+m(1-\gamma)\left|f^{(\eta+1)}\left(\frac{\mu}{m}\right)\right|\right]d\gamma\\
&\leq \frac{\mu-\sigma}{2}\left\{\left|f^{(\eta+1)}(\sigma)\right|\left(\int_0^{\frac{1}{2}}\gamma(1-2\gamma)^{\eta-\alpha}d\gamma+\int_{\frac{1}{2}}^1\gamma(2\gamma-1)^{\eta-\alpha}d\gamma\right)\right.\\
&\quad\left.+m\left|f^{(\eta+1)}\left(\frac{\mu}{m}\right)\right|\left(\int_0^{\frac{1}{2}}(1-\gamma)(1-2\gamma)^{\eta-\alpha}d\gamma+\int_{\frac{1}{2}}^1(1-\gamma)(2\gamma-1)^{\eta-\alpha}d\gamma\right)\right\}\\
&= \frac{\mu-\sigma}{4(\eta-\alpha+1)}\left(\left|f^{(\eta+1)}(\sigma)\right|+m\left|f^{(\eta+1)}\left(\frac{\mu}{m}\right)\right|\right).
\end{aligned}$$

Calculate the integrals in parentheses and multiply by their coefficients, we obtain inequality (3.1). □

Theorem 3.2. *Let $f:[\sigma,\mu]\to\mathbb{R}$ be a differentiable mapping on $[\sigma,\mu]$ with $\sigma<\mu$ and $f^{(\eta+1)}\in L[\sigma,\mu]$. If $\left|f^{(\eta+1)}\right|^{q}$ is m-convexity and $m\in(0,1]$, then the following inequality holds:*

$$\left|\frac{f^{(\eta)}(\sigma)+f^{(\eta)}(\mu)}{2}-\frac{\Gamma(\eta-\alpha+1)}{2(\mu-\sigma)^{\eta-\alpha}}\left[({}^{C}\mathscr{D}^{\alpha}_{\sigma^{+}}f)(\mu)+(-1)^{\eta}({}^{C}\mathscr{D}^{\alpha}_{\mu^{-}}f)(\sigma)\right]\right| \tag{3.2}$$

$$\leq \frac{\mu-\sigma}{2}\left(\frac{1}{(p(\eta-\alpha)+1)^{\frac{1}{p}}}\right)\left(\frac{\left|f^{(\eta+1)}(\sigma)\right|^{q}+m\left|f^{(\eta+1)}\left(\frac{\mu}{m}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}.$$

where $\alpha>0$ *and* $\alpha\notin\{1,2,3,...\}$, $\eta=[\alpha]+1$, $q>1$, $p=\frac{q}{q-1}$.

Proof. Let the left side of Lemma 1 be $\Sigma(f,\Gamma,\sigma,\mu)$. Since $\alpha\in[0,1]$ and $\forall\gamma_1,\gamma_2\in[0,1]$, $\left|\gamma_{\gamma}^{\eta-\alpha}-\gamma_2^{\eta-\alpha}\right|\leq|\gamma_1-\gamma_2|^{\eta-\alpha}$, we can write the following inequality with properties of modulus:

$$|\Sigma(f,\Gamma,\sigma,\mu)|\leq\frac{\mu-\sigma}{2}\int_0^1|1-2\gamma|^{\eta-\alpha}\left|f^{(\eta+1)}(\gamma\sigma+(1-\gamma)\mu)\right|d\gamma.$$

By applying Hölder's inequality to the right hand side of the above inequality and utilizing m-convexity of $\left|f^{(\eta+1)}\right|^{q}$, we have

$$\begin{aligned}|\Sigma(f,\Gamma,\sigma,\mu)| &\leq \frac{\mu-\sigma}{2}\int_0^1|1-2\gamma|^{\eta-\alpha}\left|f^{(\eta+1)}(\gamma\sigma+(1-\gamma)\mu)\right|d\gamma\\ &\leq \frac{\mu-\sigma}{2}\left(\int_0^1|1-2\gamma|^{p(\eta-\alpha)}d\gamma\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(\eta+1)}\left(\gamma\sigma+m(1-\gamma)\frac{\mu}{m}\right)\right|^{q}\right)^{\frac{1}{q}}\\ &\leq \frac{\mu-\sigma}{2}\left(\frac{1}{p(\eta-\alpha)+1}\right)^{\frac{1}{p}}\left(\frac{\left|f^{(\eta+1)}(\sigma)\right|^{q}+m\left|f^{(\eta+1)}\left(\frac{\mu}{m}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

This completes the proof of inequality (3.2). Here it can be easily checked that

$$\begin{aligned}\left(\int_0^1|1-2\gamma|^{p(\eta-\alpha)}d\gamma\right)^{\frac{1}{p}} &= \frac{1}{(p(\eta-\alpha)+1)^{\frac{1}{p}}},\\ \left|f^{(\eta+1)}(\sigma)\right|^{q}\int_0^1\gamma d\gamma &= \frac{\left|f^{(\eta+1)}(\sigma)\right|^{q}}{2},\\ m\left|f^{(\eta+1)}\left(\frac{\mu}{m}\right)\right|^{q}\int_0^1(1-\gamma)d\gamma &= m\frac{\left|f^{(\eta+1)}\left(\frac{\mu}{m}\right)\right|^{q}}{2}.\end{aligned}$$

□

Theorem 3.3. *Let* $f:\mathscr{I}\subset\mathbb{R}\to\mathbb{R}$, $\mathscr{I}\subset[0,\infty)$, *be a differentiable function on* $\mathscr{I}$ *such that* $f^{(\eta+1)}\in L[\sigma,\mu]$ *with* $\sigma\leq\varkappa<\vartheta\leq\mu$, $\gamma\in[0,1]$. *If* $f^{(\eta+1)}$ *is m-convex function, for all* $\alpha>0$ *and* $m\in(0,1]$, *then*

$$\frac{1}{\vartheta-\varkappa}f^{(\eta)}(\vartheta)-\frac{(-1)^{\eta}\Gamma(\eta-\alpha+1)}{(\vartheta-\varkappa)^{\eta-\alpha+1}}({}^{C}\mathscr{D}_{\vartheta-}^{\alpha}f)(\varkappa) \tag{3.3}$$

$$\leq f(\varkappa)\frac{\eta-\alpha}{\eta-\alpha+2}\beta(2,\eta-\alpha)+mf\left(\frac{\vartheta}{m}\right)\frac{1}{2(\eta-\alpha)+1}.$$

Proof. From Lemma 1.9 and m-convexity of $f^{(\eta+1)}$, we have

$$\begin{aligned}&\frac{1}{\vartheta-\varkappa}f^{(\eta)}(\vartheta)-\frac{(-1)^{\eta}\Gamma(\eta-\alpha+1)}{(\vartheta-\varkappa)^{\eta-\alpha+1}}({}^{C}\mathscr{D}_{\vartheta-}^{\alpha}f)(\varkappa)\\ &= \int_0^1(1-\gamma)^{\eta-\alpha}f^{(\eta+1)}\left(\gamma\varkappa+m(1-\gamma)\frac{\vartheta}{m}\right)d\gamma\end{aligned}$$

$$
\begin{aligned}
&\leq f(\varkappa)\int_0^1 \gamma(1-\gamma)^{\eta-\alpha}d\gamma + mf\left(\frac{\vartheta}{m}\right)\int_0^1 (1-\gamma)^{2(\eta-\alpha)}d\gamma \\
&= f(\varkappa)\beta(2,\eta-\alpha+1) + mf\left(\frac{\vartheta}{m}\right)\frac{1}{\eta-\alpha+1} \\
&= f(\varkappa)\frac{\eta-\alpha}{\eta-\alpha+2}\beta(2,\eta-\alpha) + mf\left(\frac{\vartheta}{m}\right)\frac{1}{2(\eta-\alpha)+1}
\end{aligned}
$$

which gives the required inequality (3.3). Here we used the property of the known function β:

$$\beta(2,\eta-\alpha+1) = \frac{\eta-\alpha}{\eta-\alpha+2}\beta(2,\eta-\alpha).$$

Corollary 3.4. *If we choose $\varkappa=\sigma$, $\vartheta=\mu$ and $m=1$ in (3.3), we have the following inequality*

$$
\begin{aligned}
&\frac{1}{\mu-\sigma}f^{(\eta)}(\mu) - \frac{(-1)^{\eta}\Gamma(\eta-\alpha+1)}{(\mu-\sigma)^{\eta-\alpha+1}}\left({}^{C}\mathscr{D}^{\alpha}_{\mu^-}f\right)(\sigma) \\
&\leq f(\sigma)\frac{\eta-\alpha}{\eta-\alpha+2}\beta(2,\eta-\alpha) + f(\mu)\frac{1}{2(\eta-\alpha)+1}.
\end{aligned}
$$

□

Theorem 3.5. *Let $\alpha>0$, $f:\mathscr{I}\subset\mathbb{R}\to\mathbb{R}$, $\mathscr{I}\subset[0,\infty)$, be a differentiable function on $\mathscr{I}$ such that $f^{(\eta+1)}\in L[\sigma,\mu]$ with $\sigma\leq\varkappa<\vartheta\leq\mu$, $\gamma\in[0,1]$. If $\left|f^{(\eta+1)}\right|^q$ is m-convexity with $q>1$, $p=\frac{q}{q-1}$ and $m\in(0,1]$, then*

$$
\begin{aligned}
&\left|\frac{1}{\vartheta-\varkappa}f^{(\eta)}(\vartheta) - \frac{(-1)^{\eta}\Gamma(\eta-\alpha+1)}{(\vartheta-\varkappa)^{\eta-\alpha+1}}\left({}^{C}\mathscr{D}^{\alpha}_{\vartheta^-}f\right)(\varkappa)\right| \\
&\leq \frac{1}{(\eta-\alpha+1)^{\frac{1}{p}}}\left[\left|f^{(\eta+1)}(\varkappa)\right|^q\beta(2,\eta-\alpha+1) + m\left|f^{(\eta+1)}\left(\frac{\vartheta}{m}\right)\right|^q\frac{1}{2(\eta-\alpha)+1}\right]^{\frac{1}{q}}.
\end{aligned}
\tag{3.4}
$$

Proof. Firstly, from Lemma 1.9, properties of modulus and power mean inequality, we get

$$
\begin{aligned}
&\left|\frac{1}{\vartheta-\varkappa}f^{(\eta)}(\vartheta) - \frac{(-1)^{\eta}\Gamma(\eta-\alpha+1)}{(\vartheta-\varkappa)^{\eta-\alpha+1}}\left({}^{C}\mathscr{D}^{\alpha}_{\vartheta^-}f\right)(\varkappa)\right| \\
&\leq \int_0^1 (1-\gamma)^{\eta-\alpha}\left|f^{(\eta+1)}\left(\gamma\varkappa+m(1-\gamma)\frac{\vartheta}{m}\right)\right|d\gamma \\
&\leq \left(\int_0^1 (1-\gamma)^{\eta-\alpha}d\gamma\right)^{\frac{1}{p}}\left[\int_0^1 (1-\gamma)^{\eta-\alpha}\left|f^{(\eta+1)}\left(\gamma\varkappa+m(1-\gamma)\frac{\vartheta}{m}\right)\right|^q d\gamma\right]^{\frac{1}{q}}.
\end{aligned}
$$

Utilizing the m-convexity of $\left|f^{(\eta+1)}\right|^q$, we can write

$$
\begin{aligned}
&\left|\frac{1}{\vartheta-\varkappa}f^{(\eta)}(\vartheta) - \frac{(-1)^{\eta}\Gamma(\eta-\alpha+1)}{(\vartheta-\varkappa)^{\eta-\alpha+1}}\left({}^{C}\mathscr{D}^{\alpha}_{\vartheta^-}f\right)(\varkappa)\right| \\
&\leq \left(\int_0^1 (1-\gamma)^{\eta-\alpha}d\gamma\right)^{\frac{1}{p}} \\
&\quad\times\left[\left|f^{(\eta+1)}(\varkappa)\right|^q\int_0^1 \gamma(1-\gamma)^{\eta-\alpha}d\gamma + m\left|f^{(\eta+1)}\left(\frac{\vartheta}{m}\right)\right|^q\int_0^1 (1-\gamma)^{2(\eta-\alpha)}d\gamma\right]^{\frac{1}{q}}
\end{aligned}
$$

$$= \frac{1}{(\eta-\alpha+1)^{\frac{1}{p}}}\left[\left|f^{(\eta+1)}(\varkappa)\right|^{q}\beta(2,\eta-\alpha+1)+m\left|f^{(\eta+1)}\left(\frac{\vartheta}{m}\right)\right|^{q}\frac{1}{2(\eta-\alpha)+1}\right]^{\frac{1}{q}}$$

which gives the desired inequality (3.4). Here we used

$$\beta(2,\eta-\alpha+1)=\int_0^1\gamma(1-\gamma)^{\eta-\alpha}d\gamma \quad\text{and}\quad \int_0^1(1-\gamma)^{2(\eta-\alpha)}d\gamma=\frac{1}{2(\eta-\alpha)+1}.$$

□

Corollary 3.6. *If we choose $\varkappa=\sigma$, $\vartheta=\mu$ and $m=1$ in (3.4), then we can write the following inequality for Caputo fractional operator*

$$\left|\frac{1}{\mu-\sigma}f^{(\eta)}(\mu)-\frac{(-1)^{\eta}\Gamma(\eta-\alpha+1)}{(\mu-\sigma)^{\eta-\alpha+1}}\left({}^{C}\mathscr{D}^{\alpha}_{\mu^-}f\right)(\sigma)\right|$$

$$\leq \left(\frac{1}{(\eta-\alpha+1)^{\frac{1}{p}}}\right)\left(\left|f^{(\eta+1)}(\sigma)\right|^{q}\beta(2,\eta-\alpha+1)+\left|f^{(\eta+1)}(\mu)\right|^{q}\frac{1}{2(\eta-\alpha)+1}\right)^{\frac{1}{q}}.$$

4. Conclusions

It is acknowledged that a subset of the set of real numbers is characterised by an infinite number of upper bounds. However, it is important to note that the smallest upper bound of the aforementioned set is unique. In the context of optimization theory, the objective is to identify the supremum of the upper bounds. It is evident that inequalities involving both right-sided and left-sided FC derivatives of non-integer order offer novel estimations for integral inequalities under convex functions. In consideration of results in this paper, researchers operating within this domain are capable of formulating the aforementioned theorems with regard to Riemann-Liouville derivatives.

Use of AI Tools

AI tools were not employed in generating, analyzing, or interpreting the results.

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