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Research Article

Tensorial and Hadamard Product Inequalities for Selfadjoint Operators in Hilbert Spaces via Two Tominaga's Results

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Abstract

In this paper we provide several generalizations for tensorial and Hadamard products of positive linear operators on complex Hilbert spaces of the celebrated scalar inequalities due to Tominaga. They give both multiplicative and additive reverses of Young's inequality for positive operators in terms of Specht's ratio and logarithmic mean.

Keywords: Tensorial product, Hadamard Product, Selfadjoint operators, Convex functions

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1. Introduction

In this paper we investigate some operator inequalities for tensorial and Hadamard products of positive operators in Hilbert spaces and obtain, in particular, generalizations of two scalar inequalities due to Tominaga that gave reverses of the celebrated Young's inequality between the arithmetic and geometric means. These achievements were accomplished by making use of the multivariate functional calculus introduced recently by H. Araki and F. Hansen in [2].

As is known to all, the famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called ν-weighted arithmetic-geometric mean inequality.

We recall that *Specht's ratio* is defined by

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

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Tominaga [12] had proved a multiplicative reverse Young inequality with the Specht's ratio [9] as follows:

$$(1-\nu)a+\nu b \leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \tag{1.2}$$

for a, $b>0$ and $\nu \in [0,1]$.

He also obtained the following additive reverse

$$(1-\nu)a+\nu b-a^{1-\nu}b^{\nu} \leq L(a,b)\ln S\left(\frac{a}{b}\right) \tag{1.3}$$

for a, $b>0$ and $\nu \in [0,1]$, where $L(\cdot,\cdot)$ is the *logarithmic mean* defined by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b-\ln a} \text{ for } b \neq a, \\ \\ a \text{ if } b=a. \end{cases}$$

If $0<m\leq a$, $b\leq M$, then also [12]

$$\left(a^{1-\nu}b^{\nu} \leq\right)(1-\nu)a+\nu b \leq S\left(\frac{M}{m}\right)a^{1-\nu}b^{\nu} \tag{1.4}$$

and

$$(0\leq)(1-\nu)a+\nu b-a^{1-\nu}b^{\nu} \leq aL\left(1,\frac{M}{m}\right)\ln S\left(\frac{M}{m}\right) \tag{1.5}$$

for $\nu \in [0,1]$.

Let $I_1,...,I_k$ be intervals from $\mathbb{R}$ and let $f: I_1\times ...\times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A=(A_1,...,A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1,...,H_k$ such that the spectrum of A_i is contained in I_i for $i=1,...,k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i=1,...,k$; by following [2], we define

$$f(A_1,...,A_k) := \int_{I_1}...\int_{I_k} f(\lambda_1,...,\lambda_k)\,dE_1(\lambda_1)\otimes ...\otimes dE_k(\lambda_k) \tag{1.6}$$

as a bounded selfadjoint operator on the tensorial product $H_1\otimes ...\otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [5] for functions of two variables and have the property that

$$f(A_1,...,A_k) = f_1(A_1)\otimes ...\otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1,...,t_k)=f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, namely

$$f(st) \geq (\leq) f(s)f(t) \text{ for all } s,t\in [0,\infty)$$

and if f is continuous on $[0,\infty)$, then [7, p. 173]

$$f(A\otimes B) \geq (\leq) f(A)\otimes f(B) \text{ for all } A,\ B\geq 0.$$

This follows by observing that, if

$$A=\int_{[0,\infty)} t\,dE(t) \text{ and } B=\int_{[0,\infty)} s\,dF(s)$$

are the spectral resolutions of A and B, then

$$f(A\otimes B)=\int_{[0,\infty)}\int_{[0,\infty)}f(st)\,dE(t)\otimes dF(s)$$

for the continuous function f on $[0,\infty)$.

Recall the *geometric operator mean* for the positive operators $A, B>0$

$$A\#_t B:=A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t\in[0,1]$ and

$$A\#B:=A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

By the definitions of # and $\otimes$ we have

$$A\#B=B\#A \text{ and } (A\#B)\otimes(B\#A)=(A\otimes B)\#(B\otimes A).$$

In 2007, S. Wada [13] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\#B)\otimes(A\#B)&\le\frac{1}{2}\left[(A\#_\alpha B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_\alpha B)\right]\\&\le\frac{1}{2}(A\otimes B+B\otimes A)\end{aligned}$$

for $A, B>0$ and $\alpha\in[0,1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A\circ B\in B(H)$ satisfying

$$\left\langle (A\circ B)e_j,e_j\right\rangle=\left\langle Ae_j,e_j\right\rangle\left\langle Be_j,e_j\right\rangle$$

for all $j\in\mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [6], we have the representation

$$A\circ B=\mathcal{U}^*(A\otimes B)\mathcal{U}, \tag{1.7}$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If f *is super-multiplicative operator concave (sub-multiplicative operator convex)* on $[0,\infty)$, then also [7, p. 173]

$$f(A\circ B)\ge(\le)\,f(A)\circ f(B) \text{ for all } A,\ B\ge 0.$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2}\le\left(\frac{A+B}{2}\right)\circ 1 \text{ for } A,\ B\ge 0$$

and *Fiedler inequality*

$$A\circ A^{-1}\ge 1 \text{ for } A>0.$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A\circ B\le\left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2} \text{ for } A,\ B\ge 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A\circ B\le\left(A^2\circ B^2\right)^{1/2} \text{ for } A,\ B\ge 0.$$

It has been shown in [8] that $\left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2}$ and $\left(A^2\circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

For some recent tensorial and Hadamard product inequalities, see [4], [10], [11] and the references therein.

Motivated by the above results, in this paper we show among others that, if the selfadjoint operators A and B satisfy the condition $0 < m \leq A$, $B \leq M$ for some constants m and M, then for $\nu \in [0,1]$

$$A^{1-\nu} \otimes B^{\nu} \leq (1-\nu) A \otimes 1 + \nu 1 \otimes B \leq S\left(\frac{M}{m}\right) A^{1-\nu} \otimes B^{\nu}$$

and

$$\begin{aligned} 0 &\leq (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \\ &\leq L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) A \otimes 1, \end{aligned}$$

where $S(\cdot)$ is the *Specht's ratio* and $L(\cdot,\cdot)$ is the logarithmic mean. We also have the following inequalities for the Hadamard product

$$A^{1-\nu} \circ B^{\nu} \leq [(1-\nu) A + \nu B] \circ 1 \leq S\left(\frac{M}{m}\right) A^{1-\nu} \circ B^{\nu}$$

and

$$0 \leq [(1-\nu) A + \nu B] \circ 1 - A^{1-\nu} \circ B^{\nu} \leq L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) A \circ 1,$$

where $\nu \in [0,1]$.

2. Main Results

Our first main result is as follows:

Theorem 2.1. *Assume that the selfadjoint operators A and B satisfy the condition $0 < m \leq A$, $B \leq M$ for some constants m and M, then for $\nu \in [0,1]$,*

$$A^{1-\nu} \otimes B^{\nu} \leq (1-\nu) A \otimes 1 + \nu 1 \otimes B \leq S\left(\frac{M}{m}\right) A^{1-\nu} \otimes B^{\nu} \tag{2.1}$$

and

$$0 \leq (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \leq L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) A \otimes 1. \tag{2.2}$$

In particular,

$$A^{1/2} \otimes B^{1/2} \leq \frac{1}{2}(A \otimes 1 + 1 \otimes B) \leq S\left(\frac{M}{m}\right) A^{1/2} \otimes B^{1/2}$$

and

$$0 \leq \frac{1}{2}(A \otimes 1 + 1 \otimes B) - A^{1/2} \otimes B^{1/2} \leq L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) A \otimes 1.$$

Proof. From (1.4) we get

$$t^{1-\nu} s^{\nu} \leq (1-\nu) t + \nu s \leq S\left(\frac{M}{m}\right) t^{1-\nu} s^{\nu} \tag{2.3}$$

for all t, $s \in [m,M]$ and $\nu \in [0,1]$.

Assume that

$$A = \int_m^M t dE(t) \text{ and } B = \int_m^M s dF(s)$$

are the spectral resolutions of A and B. Now, if we take the double integral $\int_m^M \int_m^M$ over $dE(t) \otimes dF(s)$ in (2.3), then we get

$$\int_m^M \int_m^M t^{1-\nu} s^{\nu} dE(t) \otimes dF(s) \leq \int_m^M \int_m^M [(1-\nu) t + \nu s] dE(t) \otimes dF(s) \tag{2.4}$$

$$\leq S\left(\frac{M}{m}\right)\int_m^M\int_m^M t^{1-\nu}s^{\nu}dE(t)\otimes dF(s).$$

Since

$$\int_m^M\int_m^M t^{1-\nu}s^{\nu}dE(t)\otimes dF(s)=A^{1-\nu}\otimes B^{\nu}$$

and

$$\int_m^M\int_m^M [(1-\nu)t+\nu s]\,dE(t)\otimes dF(s)=(1-\nu)A\otimes 1+\nu 1\otimes B,$$

then, by (2.4) we get (2.1).

By (1.5) we get

$$\begin{aligned}0&\leq\int_m^M\int_m^M [(1-\nu)t+\nu s]\,dE(t)\otimes dF(s)\\&-\int_m^M\int_m^M t^{1-\nu}s^{\nu}dE(t)\otimes dF(s)\\&\leq L\left(1,\frac{M}{m}\right)\ln S\left(\frac{M}{m}\right)\int_m^M\int_m^M t\,dE(t)\otimes dF(s)\end{aligned}\tag{2.5}$$

and since

$$\int_m^M\int_m^M t\,dE(t)\otimes dF(s)=A\otimes 1,$$

hence by (2.5) we deduce (2.2). □

Remark 2.2. If $0<m\leq A\leq M$ for some constants m and M, then

$$A^{1-\nu}\otimes A^{\nu}\leq(1-\nu)A\otimes 1+\nu 1\otimes A\leq S\left(\frac{M}{m}\right)A^{1-\nu}\otimes A^{\nu}$$

and

$$0\leq(1-\nu)A\otimes 1+\nu 1\otimes A-A^{1-\nu}\otimes A^{\nu}\leq L\left(1,\frac{M}{m}\right)\ln S\left(\frac{M}{m}\right)A\otimes 1.$$

In particular,

$$A^{1/2}\otimes A^{1/2}\leq\frac{1}{2}(A\otimes 1+1\otimes A)\leq S\left(\frac{M}{m}\right)A^{1/2}\otimes A^{1/2}$$

and

$$0\leq\frac{1}{2}(A\otimes 1+1\otimes A)-A^{1/2}\otimes A^{1/2}\leq L\left(1,\frac{M}{m}\right)\ln S\left(\frac{M}{m}\right)A\otimes 1.$$

Corollary 2.3. *Assume that the selfadjoint operators A_i and B_i satisfy the condition $0<m\leq A_i,\,B_i\leq M$, $p_i\geq 0$ for $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$, then*

$$\begin{aligned}&\left(\sum_{i=1}^n p_iA_i^{1-\nu}\right)\otimes\left(\sum_{i=1}^n p_iB_i^{\nu}\right)\\&\leq(1-\nu)\left(\sum_{i=1}^n p_iA_i\right)\otimes 1+\nu 1\otimes\left(\sum_{i=1}^n p_iB_i\right)\\&\leq S\left(\frac{M}{m}\right)\left(\sum_{i=1}^n p_iA_i^{1-\nu}\right)\otimes\left(\sum_{i=1}^n p_iB_i^{\nu}\right)\end{aligned}\tag{2.6}$$

and

$$0 \le (1-\nu)\left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1 + \nu 1 \otimes \left(\sum_{i=1}^{n} p_i B_i\right) \tag{2.7}$$
$$- \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \otimes \left(\sum_{i=1}^{n} p_i B_i^{\nu}\right)$$
$$\le L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) \left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1.$$

Proof. From (2.1) we have

$$A_i^{1-\nu} \otimes B_j^{\nu} \le (1-\nu) A_i \otimes 1 + \nu 1 \otimes B_j \le S\left(\frac{M}{m}\right) A_i^{1-\nu} \otimes B_j^{\nu}$$

for $i,\ j \in \{1,...,n\}$.

If we multiply by $p_i p_j \ge 0$ and sum, then we get

$$\sum_{i,j=1}^{n} p_i p_j A_i^{1-\nu} \otimes B_j^{\nu} \le \sum_{i,j=1}^{n} p_i p_j \left[(1-\nu) A_i \otimes 1 + \nu 1 \otimes B_j\right]$$
$$\le S\left(\frac{M}{m}\right) \sum_{i,j=1}^{n} p_i p_j A_i^{1-\nu} \otimes B_j^{\nu},$$

which is equivalent to (2.6).

The inequality (2.7) follows in a similar way from (2.2). □

Remark 2.4. If we take $B_i = A_i$, $i \in \{1,...,n\}$ in Corollary 2.3, then we get

$$\left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{\nu}\right)$$
$$\le (1-\nu)\left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1 + \nu 1 \otimes \left(\sum_{i=1}^{n} p_i A_i\right)$$
$$\le S\left(\frac{M}{m}\right) \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{\nu}\right)$$

and

$$0 \le (1-\nu)\left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1 + \nu 1 \otimes \left(\sum_{i=1}^{n} p_i A_i\right)$$
$$- \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{\nu}\right)$$
$$\le L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) \left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1.$$

Corollary 2.5. *With the assumptions of Theorem 2.1,*

$$A^{1-\nu}\circ B^{\nu}\leq[(1-\nu)A+\nu B]\circ 1\leq S\left(\frac{M}{m}\right)A^{1-\nu}\circ B^{\nu} \tag{2.8}$$

and

$$0\leq[(1-\nu)A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu}\leq L\left(1,\frac{M}{m}\right)\ln S\left(\frac{M}{m}\right)A\circ 1. \tag{2.9}$$

In particular,

$$A^{1/2}\circ B^{1/2}\leq\frac{A+B}{2}\circ 1\leq S\left(\frac{M}{m}\right)A^{1/2}\circ B^{1/2}$$

and

$$0\leq\frac{A+B}{2}\circ 1-A^{1/2}\circ B^{1/2}\leq L\left(1,\frac{M}{m}\right)\ln S\left(\frac{M}{m}\right)A\circ 1.$$

Proof. If we use the identity (1.7) and apply $\mathscr{U}^*$ to the left and $\mathscr{U}$ to the right of inequality (2.1), we get

$$\begin{aligned}\mathscr{U}^*\left(A^{1-\nu}\otimes B^{\nu}\right)\mathscr{U}&\leq\mathscr{U}^*\left[(1-\nu)A\otimes 1+\nu 1\otimes B\right]\mathscr{U} \\ &\leq S\left(\frac{M}{m}\right)\mathscr{U}^*\left(A^{1-\nu}\otimes B^{\nu}\right)\mathscr{U},\end{aligned} \tag{2.10}$$

which is equivalent to (2.8). □

Remark 2.6. Assume that the selfadjoint operators A_i and B_i satisfy the condition $0<m\leq A_i,\ B_i\leq M,\ p_i\geq 0$ for $i\in\{1,...,n\}$ with $\sum_{i=1}^{n}p_i=1$, then by Remark 2.4 we get

$$\begin{aligned}\left(\sum_{i=1}^{n}p_iA_i^{1-\nu}\right)\circ\left(\sum_{i=1}^{n}p_iA_i^{\nu}\right)&\leq\left(\sum_{i=1}^{n}p_iA_i\right)\circ 1 \\ &\leq S\left(\frac{M}{m}\right)\left(\sum_{i=1}^{n}p_iA_i^{1-\nu}\right)\circ\left(\sum_{i=1}^{n}p_iA_i^{\nu}\right)\end{aligned}$$

and

$$\begin{aligned}0&\leq\left(\sum_{i=1}^{n}p_iA_i\right)\circ 1-\left(\sum_{i=1}^{n}p_iA_i^{1-\nu}\right)\circ\left(\sum_{i=1}^{n}p_iA_i^{\nu}\right) \\ &\leq L\left(1,\frac{M}{m}\right)\ln S\left(\frac{M}{m}\right)\left(\sum_{i=1}^{n}p_iA_i\right)\circ 1.\end{aligned}$$

Theorem 2.7. *Assume that f, g are continuous and nonnegative on the interval I and there exists $0\leq\gamma<\Gamma$ such that*

$$\gamma\leq\frac{f(t)}{g(t)}\leq\Gamma\ \textit{for all}\ t\in I,$$

then for the selfadjoint operators A and B with spectra $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset I$,

$$\begin{aligned}&\left(f^{2(1-\nu)}(A)\,g^{2\nu}(A)\right)\otimes\left(f^{2\nu}(B)\,g^{2(1-\nu)}(B)\right) \\ &\leq(1-\nu)f^2(A)\otimes g^2(B)+\nu g^2(A)\otimes f^2(B) \\ &\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)\left(f^{2(1-\nu)}(A)\,g^{2\nu}(A)\right)\otimes\left(f^{2\nu}(B)\,g^{2(1-\nu)}(B)\right)\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}0 &\leq (1-\nu) f^2(A) \otimes g^2(B) + \nu g^2(A) \otimes f^2(B) \\ &\quad - \left(f^{2(1-\nu)}(A) g^{2\nu}(A)\right) \otimes \left(f^{2\nu}(B) g^{2(1-\nu)}(B)\right) \\ &\leq L\left(1, \left(\frac{\Gamma}{\gamma}\right)^2\right) \ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) f^2(A) \otimes g^2(B).\end{aligned} \tag{2.12}$$

In particular, for $\nu = 1/2$,

$$\begin{aligned}(f(A) g(A)) \otimes (f(B) g(B)) &\leq \frac{1}{2}\left[f^2(A) \otimes g^2(B) + g^2(A) \otimes f^2(B)\right] \\ &\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)(f(A) g(A)) \otimes (f(B) g(B))\end{aligned}$$

and

$$\begin{aligned}0 &\leq \frac{1}{2}\left[f^2(A) \otimes g^2(B) + g^2(A) \otimes f^2(B)\right] - (f(A) g(A)) \otimes (f(B) g(B)) \\ &\leq L\left(1, \left(\frac{\Gamma}{\gamma}\right)^2\right) \ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) f^2(A) \otimes g^2(B).\end{aligned}$$

Proof. For any $t, s \in I$ we have

$$\gamma^2 \leq \frac{f^2(t)}{g^2(t)}, \frac{f^2(s)}{g^2(s)} \leq \Gamma^2.$$

If we use the inequality (1.4) for

$$a = \frac{f^2(t)}{g^2(t)}, \ b = \frac{f^2(s)}{g^2(s)},$$

then we get

$$\begin{aligned}\left(\frac{f^2(t)}{g^2(t)}\right)^{1-\nu}\left(\frac{f^2(s)}{g^2(s)}\right)^{\nu} &\leq (1-\nu)\frac{f^2(t)}{g^2(t)} + \nu\frac{f^2(s)}{g^2(s)} \\ &\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)\left(\frac{f^2(t)}{g^2(t)}\right)^{1-\nu}\left(\frac{f^2(s)}{g^2(s)}\right)^{\nu}\end{aligned} \tag{2.13}$$

for any $t, s \in I$.

Now, if we multiply (2.13) by $g^2(t) g^2(s) > 0$, then we get

$$\begin{aligned}&f^{2(1-\nu)}(t) g^{2\nu}(t) f^{2\nu}(s) g^{2(1-\nu)}(s) \\ &\leq (1-\nu) f^2(t) g^2(s) + \nu g^2(t) f^2(s) \\ &\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) f^{2(1-\nu)}(t) g^{2\nu}(t) f^{2\nu}(s) g^{2(1-\nu)}(s)\end{aligned} \tag{2.14}$$

for any $t, s \in I$.

Assume that

$$A = \int_I t dE(t) \text{ and } B = \int_I s dF(s)$$

are the spectral resolutions of A and B.

Further on, if we take the double integral $\int_I \int_I$ over $dE(t) \otimes dF(s)$ in (2.14), then we get

$$\begin{aligned}
&\int_I \int_I f^{2(1-\nu)}(t) g^{2\nu}(t) f^{2\nu}(s) g^{2(1-\nu)}(s) dE(t) \otimes dF(s) \\
&\leq (1-\nu) \int_I \int_I f^2(t) g^2(s) dE(t) \otimes dF(s) \\
&+ \nu \int_I \int_I g^2(t) f^2(s) dE(t) \otimes dF(s) \\
&\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \int_I \int_I f^{2(1-\nu)}(t) g^{2\nu}(t) f^{2\nu}(s) g^{2(1-\nu)}(s) dE(t) \otimes dF(s).
\end{aligned} \tag{2.15}$$

Since

$$\begin{aligned}
&\int_I \int_I f^{2(1-\nu)}(t) g^{2\nu}(t) f^{2\nu}(s) g^{2(1-\nu)}(s) dE(t) \otimes dF(s) \\
&= \int_I f^{2(1-\nu)}(t) g^{2\nu}(t) dE(t) \otimes \int_I f^{2\nu}(s) g^{2(1-\nu)}(s) dF(s) \\
&= \left(f^{2(1-\nu)}(A) g^{2\nu}(A)\right) \otimes \left(f^{2\nu}(B) g^{2(1-\nu)}(B)\right),
\end{aligned}$$

$$\begin{aligned}
\int_I \int_I f^2(t) g^2(s) dE(t) \otimes dF(s) &= \int_I f^2(t) dE(t) \otimes \int_I g^2(s) dF(s) \\
&= f^2(A) \otimes g^2(B),
\end{aligned}$$

and

$$\begin{aligned}
\int_I \int_I g^2(t) f^2(s) dE(t) \otimes dF(s) &= \int_I g^2(t) dE(t) \otimes \int_I f^2(s) dF(s) \\
&= g^2(A) \otimes f^2(B),
\end{aligned}$$

hence by (2.15) we get (2.11).

From (1.5) we obtain

$$\begin{aligned}
&(0 \leq)\, (1-\nu) \frac{f^2(t)}{g^2(t)} + \nu \frac{f^2(s)}{g^2(s)} - \left(\frac{f^2(t)}{g^2(t)}\right)^{1-\nu} \left(\frac{f^2(s)}{g^2(s)}\right)^{\nu} \\
&\leq \frac{f^2(t)}{g^2(t)} L\left(1, \left(\frac{\Gamma}{\gamma}\right)^2\right) \ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)
\end{aligned}$$

for any $t, s \in I$.

If we multiply by $g^2(t) g^2(s) > 0$ then we get

$$\begin{aligned}
&(1-\nu) f^2(t) g^2(s) + \nu g^2(t) f^2(s) - f^{2(1-\nu)}(t) g^{2\nu}(t) f^{2\nu}(s) g^{2(1-\nu)}(s) \\
&\leq f^2(t) g^2(s) L\left(1, \left(\frac{\Gamma}{\gamma}\right)^2\right) \ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)
\end{aligned} \tag{2.16}$$

for any $t, s \in I$.

If we take the double integral $\int_I \int_I$ over $dE(t) \otimes dF(s)$ in (2.16) and use a similar argument as above, we deduce the desired result (2.12). $\square$

Corollary 2.8. *With the assumption of Theorem 2.7, we have the following inequalities for the Hadamard product*

$$\begin{aligned}&\left(f^{2(1-\nu)}(A)\,g^{2\nu}(A)\right)\circ\left(f^{2\nu}(B)\,g^{2(1-\nu)}(B)\right)\\&\leq(1-\nu)\,f^{2}(A)\circ g^{2}(B)+\nu g^{2}(A)\circ f^{2}(B)\\&\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^{2}\right)\left(f^{2(1-\nu)}(A)\,g^{2\nu}(A)\right)\circ\left(f^{2\nu}(B)\,g^{2(1-\nu)}(B)\right)\end{aligned}$$

and

$$\begin{aligned}0&\leq(1-\nu)\,f^{2}(A)\circ g^{2}(B)+\nu g^{2}(A)\circ f^{2}(B)\\&-\left(f^{2(1-\nu)}(A)\,g^{2\nu}(A)\right)\circ\left(f^{2\nu}(B)\,g^{2(1-\nu)}(B)\right)\\&\leq L\left(1,\left(\frac{\Gamma}{\gamma}\right)^{2}\right)\ln S\left(\left(\frac{\Gamma}{\gamma}\right)^{2}\right)f^{2}(A)\circ g^{2}(B).\end{aligned}$$

In particular, for $\nu=1/2$,

$$\begin{aligned}(f(A)\,g(A))\circ(f(B)\,g(B))&\leq\frac{1}{2}\left[f^{2}(A)\circ g^{2}(B)+g^{2}(A)\circ f^{2}(B)\right]\\&\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^{2}\right)(f(A)\,g(A))\circ(f(B)\,g(B))\end{aligned}$$

and

$$\begin{aligned}0&\leq\frac{1}{2}\left[f^{2}(A)\circ g^{2}(B)+g^{2}(A)\circ f^{2}(B)\right]-(f(A)\,g(A))\circ(f(B)\,g(B))\\&\leq L\left(1,\left(\frac{\Gamma}{\gamma}\right)^{2}\right)\ln S\left(\left(\frac{\Gamma}{\gamma}\right)^{2}\right)f^{2}(A)\circ g^{2}(B).\end{aligned}$$

Remark 2.9. If we take $B=A$ in Corollary 2.8, then we get the simpler inequalities

$$\begin{aligned}&\left(f^{2(1-\nu)}(A)\,g^{2\nu}(A)\right)\circ\left(f^{2\nu}(A)\,g^{2(1-\nu)}(A)\right)\\&\leq f^{2}(A)\circ g^{2}(A)\\&\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^{2}\right)\left(f^{2(1-\nu)}(A)\,g^{2\nu}(A)\right)\circ\left(f^{2\nu}(A)\,g^{2(1-\nu)}(A)\right)\end{aligned}$$

and

$$\begin{aligned}0&\leq f^{2}(A)\circ g^{2}(A)-\left(f^{2(1-\nu)}(A)\,g^{2\nu}(A)\right)\circ\left(f^{2\nu}(A)\,g^{2(1-\nu)}(A)\right)\\&\leq L\left(1,\left(\frac{\Gamma}{\gamma}\right)^{2}\right)\ln S\left(\left(\frac{\Gamma}{\gamma}\right)^{2}\right)f^{2}(A)\circ g^{2}(A).\end{aligned}$$

In particular, for $\nu=1/2$,

$$(f(A)\,g(A))\circ(f(A)\,g(A))\leq f^{2}(A)\circ g^{2}(A)$$

$$\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)(f(A)g(A))\circ(f(A)g(A))$$

and

$$0 \leq f^2(A)\circ g^2(A)-(f(A)g(A))\circ(f(A)g(A))$$
$$\leq L\left(1,\left(\frac{\Gamma}{\gamma}\right)^2\right)\ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)f^2(A)\circ g^2(A).$$

Corollary 2.10. *With the assumption of Theorem 2.7, then for the selfadjoint operators A_i and B_i with spectra $\operatorname{Sp}(A_i)$, $\operatorname{Sp}(B_i)\subset I$, and $p_i\geq 0$ for $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$, then*

$$\left(\sum_{i=1}^n p_i f^{2(1-\nu)}(A_i)g^{2\nu}(A_i)\right)\otimes\left(\sum_{i=1}^n p_i f^{2\nu}(B_i)g^{2(1-\nu)}(B_i)\right)$$
$$\leq(1-\nu)\left(\sum_{i=1}^n p_i f^2(A_i)\right)\otimes\left(\sum_{i=1}^n p_i g^2(B_i)\right)$$
$$+\nu\left(\sum_{i=1}^n p_i g^2(A_i)\right)\otimes\left(\sum_{i=1}^n p_i f^2(B_i)\right)$$
$$\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)\left(\sum_{i=1}^n p_i f^{2(1-\nu)}(A_i)g^{2\nu}(A_i)\right)$$
$$\otimes\left(\sum_{i=1}^n p_i f^{2\nu}(B_i)g^{2(1-\nu)}(B_i)\right)$$

and

$$0\leq(1-\nu)\left(\sum_{i=1}^n p_i f^2(A_i)\right)\otimes\left(\sum_{i=1}^n p_i g^2(B_i)\right)$$
$$+\nu\left(\sum_{i=1}^n p_i g^2(A_i)\right)\otimes\left(\sum_{i=1}^n p_i f^2(B_i)\right)$$
$$-\left(\sum_{i=1}^n p_i f^{2(1-\nu)}(A_i)g^{2\nu}(A_i)\right)\otimes\left(\sum_{i=1}^n p_i f^{2\nu}(B_i)g^{2(1-\nu)}(B_i)\right)$$
$$\leq L\left(1,\left(\frac{\Gamma}{\gamma}\right)^2\right)\ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)\left(\sum_{i=1}^n p_i f^2(A_i)\right)\otimes\left(\sum_{i=1}^n p_i g^2(B_i)\right).$$

If take $B_i=A_i$ and consider the Hadamard product version, then we get the simpler inequalities

$$\left(\sum_{i=1}^n p_i f^{2(1-\nu)}(A_i)g^{2\nu}(A_i)\right)\circ\left(\sum_{i=1}^n p_i f^{2\nu}(A_i)g^{2(1-\nu)}(A_i)\right) \tag{2.17}$$
$$\leq\left(\sum_{i=1}^n p_i f^2(A_i)\right)\circ\left(\sum_{i=1}^n p_i g^2(A_i)\right)$$
$$\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)\left(\sum_{i=1}^n p_i f^{2(1-\nu)}(A_i)g^{2\nu}(A_i)\right)$$

$$\circ\left(\sum_{i=1}^{n} p_i f^{2\nu}(A_i)\, g^{2(1-\nu)}(A_i)\right)$$

and

$$\begin{aligned}0 &\leq \left(\sum_{i=1}^{n} p_i f^{2}(A_i)\right)\circ\left(\sum_{i=1}^{n} p_i g^{2}(A_i)\right) \\ &-\left(\sum_{i=1}^{n} p_i f^{2(1-\nu)}(A_i)\, g^{2\nu}(A_i)\right)\circ\left(\sum_{i=1}^{n} p_i f^{2\nu}(A_i)\, g^{2(1-\nu)}(A_i)\right) \\ &\leq L\left(1,\left(\frac{\Gamma}{\gamma}\right)^{2}\right)\ln S\left(\left(\frac{\Gamma}{\gamma}\right)^{2}\right)\left(\sum_{i=1}^{n} p_i f^{2}(A_i)\right)\circ\left(\sum_{i=1}^{n} p_i g^{2}(A_i)\right).\end{aligned} \tag{2.18}$$

3. Some Related Results

Further on, observe that if a, $b>0$ and

$$0<l^{-1}\leq\frac{a}{b}\leq L<\infty,$$

for some L, $l>0$ with $Ll>1$, then

$$S\left(\frac{a}{b}\right)\leq\max\left\{S\left(l^{-1}\right),S(L)\right\}=\max\left\{S(l),S(L)\right\}$$

and by (1.2) we have

$$(1-\nu)a+\nu b\leq\max\left\{S(l),S(L)\right\}a^{1-\nu}b^{\nu} \tag{3.1}$$

for every $\nu\in[0,1]$.

Theorem 3.1. *Assume that*

$$0<m_1\leq f(t)\leq M_1<\infty,\ 0<m_2\leq g(t)\leq M_2<\infty, \tag{3.2}$$

for $t\in I$. If $u(t)$, $v(t)\geq 0$ and continuous on I, then for the selfadjoint operators A and B with spectra $\operatorname{Sp}(A)$, $\operatorname{Sp}(B)\subset I$,

$$\begin{aligned}&(f(A)\,u(A))\otimes(v(B)\,g(B)) \\ &\leq\frac{1}{p}\left(u(A)\,f^{p}(A)\right)\otimes v(B)+\frac{1}{q}u(A)\otimes\left(g^{q}(B)\,v(B)\right) \\ &\leq\max\left\{S\left(\frac{M_2^{q}}{m_1^{p}}\right),S\left(\frac{M_1^{p}}{m_2^{q}}\right)\right\}(f(A)\,u(A))\otimes(v(B)\,g(B)).\end{aligned} \tag{3.3}$$

In particular, for $p=q=2$

$$\begin{aligned}&(f(A)\,u(A))\otimes(v(B)\,g(B)) \\ &\leq\frac{1}{2}\left[\left(u(A)\,f^{2}(A)\right)\otimes v(B)+u(A)\otimes\left(g^{2}(B)\,v(B)\right)\right] \\ &\leq\max\left\{S\left(\left(\frac{M_2}{m_1}\right)^{2}\right),S\left(\left(\frac{M_1}{m_2}\right)^{2}\right)\right\}(f(A)\,u(A))\otimes(v(B)\,g(B)).\end{aligned}$$

Proof. Now, if we write the inequality (3.1) for $l = \frac{M_2^q}{m_1^p}$, $L = \frac{M_1^p}{m_2^q}$, $a = f^p(t)$, $b = g^q(s)$ and $\nu = \frac{1}{q}$, and use Young's inequality, then we get

$$f(t)g(s) \le \frac{1}{p}f^p(t) + \frac{1}{q}g^q(s) \le \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\} f(t)g(s) \tag{3.4}$$

for any $t, s \in I$.

If we multiply (3.4) by $u(t)v(s) \ge 0$, then we get

$$\begin{aligned} f(t)u(t)v(s)g(s) &\le \frac{1}{p}u(t)f^p(t)v(s) + \frac{1}{q}u(t)g^q(s)v(s) \\ &\le \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\} f(t)u(t)v(s)g(s) \end{aligned} \tag{3.5}$$

for any $t, s \in I$.

If we take the double integral $\int_I \int_I$ over $dE(t) \otimes dF(s)$ in (3.5) and use a similar argument as above, we deduce the desired result (3.3). □

Corollary 3.2. *With the assumptions of Theorem 3.1 we have the tensorial inequalities*

$$\begin{aligned} &(f(A)g(A)) \otimes (f(B)g(B)) \\ &\le \frac{1}{p}(g(A)f^p(A)) \otimes f(B) + \frac{1}{q}g(A) \otimes (g^q(B)f(B)) \\ &\le \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\}(f(A)g(A)) \otimes (f(B)g(B)) \end{aligned}$$

and

$$\begin{aligned} &\left(f^2(A)\right) \otimes \left(g^2(B)\right) \\ &\le \frac{1}{p}\left(f^{p+1}(A)\right) \otimes g(B) + \frac{1}{q}f(A) \otimes \left(g^{q+1}(B)\right) \\ &\le \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\}\left(f^2(A)\right) \otimes \left(g^2(B)\right). \end{aligned}$$

For $p = q = 2$, we obtain

$$\begin{aligned} &(f(A)g(A)) \otimes (f(B)g(B)) \\ &\le \frac{1}{2}\left[\left(g(A)f^2(A)\right) \otimes f(B) + g(A) \otimes \left(g^2(B)f(B)\right)\right] \\ &\le \max\left\{S\left(\left(\frac{M_2}{m_1}\right)^2\right), S\left(\left(\frac{M_1}{m_2}\right)^2\right)\right\}(f(A)g(A)) \otimes (f(B)g(B)) \end{aligned}$$

and

$$\begin{aligned} &\left(f^2(A)\right) \otimes \left(g^2(B)\right) \\ &\le \frac{1}{2}\left[\left(f^3(A)\right) \otimes g(B) + f(A) \otimes \left(g^3(B)\right)\right] \\ &\le \max\left\{S\left(\left(\frac{M_2}{m_1}\right)^2\right), S\left(\left(\frac{M_1}{m_2}\right)^2\right)\right\}\left(f^2(A)\right) \otimes \left(g^2(B)\right). \end{aligned}$$

We have the following results for Hadamard product:

Corollary 3.3. *With the assumptions of Theorem 3.1,*

$$\begin{aligned}&(f(A)u(A))\circ(v(B)g(B))\\&\le\frac{1}{p}(u(A)f^p(A))\circ v(B)+\frac{1}{q}u(A)\circ(g^q(B)v(B))\\&\le\max\left\{S\left(\frac{M_2^q}{m_1^p}\right),S\left(\frac{M_1^p}{m_2^q}\right)\right\}(f(A)u(A))\circ(v(B)g(B)).\end{aligned}$$

In particular, for $p=q=2$

$$\begin{aligned}&(f(A)u(A))\circ(v(B)g(B))\\&\le\frac{1}{2}\left[\left(u(A)f^2(A)\right)\circ v(B)+u(A)\circ\left(g^2(B)v(B)\right)\right]\\&\le\max\left\{S\left(\left(\frac{M_2}{m_1}\right)^2\right),S\left(\left(\frac{M_1}{m_2}\right)^2\right)\right\}(f(A)u(A))\circ(v(B)g(B)).\end{aligned}$$

For $v=u$, we get

$$\begin{aligned}&(f(A)u(A))\circ(u(B)g(B))\\&\le\frac{1}{p}(u(A)f^p(A))\circ u(B)+\frac{1}{q}u(A)\circ(g^q(B)u(B))\\&\le\max\left\{S\left(\frac{M_2^q}{m_1^p}\right),S\left(\frac{M_1^p}{m_2^q}\right)\right\}(f(A)u(A))\circ(u(B)g(B)).\end{aligned}\tag{3.6}$$

In particular, for $p=q=2$

$$\begin{aligned}&(f(A)u(A))\circ(u(B)g(B))\\&\le\frac{1}{2}\left[\left(u(A)f^2(A)\right)\circ u(B)+u(A)\circ\left(g^2(B)u(B)\right)\right]\\&\le\max\left\{S\left(\left(\frac{M_2}{m_1}\right)^2\right),S\left(\left(\frac{M_1}{m_2}\right)^2\right)\right\}(f(A)u(A))\circ(u(B)g(B)).\end{aligned}\tag{3.7}$$

Moreover, if we take in (3.6) and (3.7) $B=A$, we get

$$\begin{aligned}&(u(A)f(A))\circ(u(A)g(A))\\&\le\left(u(A)\left[\frac{1}{p}f^p(A)+\frac{1}{q}g^q(A)\right]\right)\circ u(A)\\&\le\max\left\{S\left(\frac{M_2^q}{m_1^p}\right),S\left(\frac{M_1^p}{m_2^q}\right)\right\}(u(A)f(A))\circ(u(A)g(A)).\end{aligned}$$

In particular, for $p=q=2$

$$\begin{aligned}&(u(A)f(A))\circ(u(A)g(A))\\&\le\left(u(A)\left[\frac{f^2(A)+g^2(A)}{2}\right]\right)\circ u(A)\end{aligned}\tag{3.8}$$

$$\leq \max\left\{S\left(\left(\frac{M_2}{m_1}\right)^2\right), S\left(\left(\frac{M_1}{m_2}\right)^2\right)\right\}(u(A) f(A)) \circ (u(A) g(A)).$$

Moreover, if we take $g = f$ in (3.8), then we get

$$\begin{aligned}(u(A) f(A)) \circ (u(A) f(A)) &\leq \left(u(A) f^2(A)\right) \circ u(A) \\ &\leq S\left(\left(\frac{M_1}{m_2}\right)^2\right)(u(A) f(A)) \circ (u(A) f(A)).\end{aligned}$$

We also have the following inequalities for sums:

Corollary 3.4. *Assume that f and g satisfy the conditions (3.2). If $u(t)$, $v(t) \geq 0$ and continuous on I, then for the selfadjoint operators A_i and B_i with $\operatorname{Sp}(A_i)$, $\operatorname{Sp}(B_i) \subset I$, and $p_i \geq 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$,*

$$\begin{aligned}&\left(\sum_{i=1}^n p_i f(A_i) u(A_i)\right) \otimes \left(\sum_{i=1}^n p_i v(B_i) g(B_i)\right) \\ &\leq \frac{1}{p}\left(\sum_{i=1}^n p_i u(A_i) f^p(A_i)\right) \otimes \left(\sum_{i=1}^n p_i v(B_i)\right) \\ &+ \frac{1}{q}\left(\sum_{i=1}^n p_i u(A_i)\right) \otimes \left(\sum_{i=1}^n p_i g^q(B_i) v(B_i)\right) \\ &\leq \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\} \\ &\times \left(\sum_{i=1}^n p_i f(A_i) u(A_i)\right) \otimes \left(\sum_{i=1}^n p_i v(B_i) g(B_i)\right).\end{aligned}$$

In particular, for $p = q = 2$

$$\begin{aligned}&\left(\sum_{i=1}^n p_i f(A_i) u(A_i)\right) \otimes \left(\sum_{i=1}^n p_i v(B_i) g(B_i)\right) \\ &\leq \frac{1}{2}\left[\left(\sum_{i=1}^n p_i u(A_i) f^2(A_i)\right) \otimes \left(\sum_{i=1}^n p_i v(B_i)\right)\right. \\ &\left. + \left(\sum_{i=1}^n p_i u(A_i)\right) \otimes \left(\sum_{i=1}^n p_i g^2(B_i) v(B_i)\right)\right] \\ &\leq \max\left\{S\left(\left(\frac{M_2}{m_1}\right)^2\right), S\left(\left(\frac{M_1}{m_2}\right)^2\right)\right\} \\ &\times \left(\sum_{i=1}^n p_i f(A_i) u(A_i)\right) \otimes \left(\sum_{i=1}^n p_i v(B_i) g(B_i)\right).\end{aligned} \tag{3.9}$$

From (3.9) for $g = f$, $v = u$ and $B_i = A_i$, $i \in \{1, ..., n\}$, we get the inequality for the Hadamard product

$$\left(\sum_{i=1}^n p_i u(A_i) f(A_i)\right) \circ \left(\sum_{i=1}^n p_i u(A_i) f(A_i)\right)$$

$$\leq \left(\sum_{i=1}^{n} p_i u(A_i) f^2(A_i)\right) \circ \left(\sum_{i=1}^{n} p_i u(A_i)\right)$$
$$\leq S\left(\left(\frac{M_1}{m_2}\right)^2\right) \left(\sum_{i=1}^{n} p_i f(A_i) u(A_i)\right) \circ \left(\sum_{i=1}^{n} p_i u(A_i) f(A_i)\right).$$

4. Some Examples

Consider the functions $f(t) = t^p$ and $g(t) = t^q$ for $t > 0$ and $p, q \neq 0$. If A, B are selfadjoint operators with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subseteq [m, M] \subset (0, \infty)$, then it follows that

$$\frac{f(t)}{g(t)} = t^{p-q}, \text{ for } t > 0.$$

Therefore

$$m^{p-q} \leq \frac{f(t)}{g(t)} \leq M^{p-q} \text{ for } t \in [m, M] \text{ and } p > q$$

and

$$M^{p-q} \leq \frac{f(t)}{g(t)} \leq m^{p-q} \text{ for } t \in [m, M] \text{ and } p < q.$$

From Theorem 2.7 we get for $p > q$ that

$$A^{2(1-\nu)p+2\nu q} \otimes B^{2\nu p+2(1-\nu)q}$$
$$\leq (1-\nu) A^{2p} \otimes B^{2q} + \nu A^{2q} \otimes B^{2p}$$
$$\leq S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) A^{2(1-\nu)p+2\nu q} \otimes B^{2\nu p+2(1-\nu)q}$$

and

$$0 \leq (1-\nu) A^{2p} \otimes B^{2q} + \nu A^{2q} \otimes B^{2p} - A^{2(1-\nu)p+2\nu q} \otimes B^{2\nu p+2(1-\nu)q}$$
$$\leq L\left(1, \left(\frac{M}{m}\right)^{2(p-q)}\right) \ln S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) A^{2p} \otimes B^{2q}.$$

In particular, for $\nu = 1/2$,

$$A^{p+q} \otimes B^{p+q} \leq \frac{1}{2}\left(A^{2p} \otimes B^{2q} + A^{2q} \otimes B^{2p}\right)$$
$$\leq S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) A^{p+q} \otimes B^{p+q}$$

and

$$0 \leq \frac{1}{2}\left(A^{2p} \otimes B^{2q} + A^{2q} \otimes B^{2p}\right) - A^{p+q} \otimes B^{p+q}$$
$$\leq L\left(1, \left(\frac{M}{m}\right)^{2(p-q)}\right) \ln S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) A^{2p} \otimes B^{2q}.$$

We also have the inequalities for the Hadamard product

$$A^{2(1-\nu)p+2\nu q} \circ B^{2\nu p+2(1-\nu)q} \tag{4.1}$$

$$\le (1-\nu)A^{2p}\circ B^{2q}+\nu A^{2q}\circ B^{2p}$$
$$\le S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right)A^{2(1-\nu)p+2\nu q}\circ B^{2\nu p+2(1-\nu)q}$$

and

$$0\le (1-\nu)A^{2p}\circ B^{2q}+\nu A^{2q}\circ B^{2p}-A^{2(1-\nu)p+2\nu q}\circ B^{2\nu p+2(1-\nu)q} \tag{4.2}$$
$$\le L\left(1,\left(\frac{M}{m}\right)^{2(p-q)}\right)\ln S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right)A^{2p}\circ B^{2q}.$$

In particular, for $\nu=1/2$,

$$A^{p+q}\circ B^{p+q}\le \frac{1}{2}\left(A^{2p}\circ B^{2q}+A^{2q}\circ B^{2p}\right) \tag{4.3}$$
$$\le S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right)A^{p+q}\circ B^{p+q}$$

and

$$0\le \frac{1}{2}\left(A^{2p}\circ B^{2q}+A^{2q}\circ B^{2p}\right)-A^{p+q}\circ B^{p+q} \tag{4.4}$$
$$\le L\left(1,\left(\frac{M}{m}\right)^{2(p-q)}\right)\ln S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right)A^{2p}\circ B^{2q}.$$

Moreover, if we take $B=A$ in (4.1)-(4.4), then we get

$$A^{2(1-\nu)p+2\nu q}\circ A^{2\nu p+2(1-\nu)q}\le A^{2p}\circ A^{2q}$$
$$\le S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right)A^{2(1-\nu)p+2\nu q}\circ A^{2\nu p+2(1-\nu)q}$$

and

$$0\le A^{2p}\circ A^{2q}-A^{2(1-\nu)p+2\nu q}\circ A^{2\nu p+2(1-\nu)q}$$
$$\le L\left(1,\left(\frac{M}{m}\right)^{2(p-q)}\right)\ln S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right)A^{2p}\circ A^{2q}.$$

In particular, for $\nu=1/2$,

$$A^{p+q}\circ A^{p+q}\le A^{2p}\circ A^{2q}\le S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right)A^{p+q}\circ A^{p+q}$$

and

$$0\le A^{2p}\circ A^{2q}-A^{p+q}\circ A^{p+q}$$
$$\le L\left(1,\left(\frac{M}{m}\right)^{2(p-q)}\right)\ln S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right)A^{2p}\circ A^{2q}.$$

Now, assume that $\operatorname{Sp}(A_i) \subseteq [m,M] \subset (0,\infty)$ and $p_i \geq 0$ for $i \in \{1,...,n\}$ with $\sum_{i=1}^{n} p_i = 1$, then from (2.17) and (2.18) we derive

$$\begin{aligned}
&\left(\sum_{i=1}^{n} p_i A_i^{2(1-\nu)p+2\nu q}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{2\nu p+2(1-\nu)q}\right) \\
&\leq \left(\sum_{i=1}^{n} p_i A_i^{2p}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{2q}\right) \\
&\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \left(\sum_{i=1}^{n} p_i A_i^{2(1-\nu)p+2\nu q}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{2\nu p+2(1-\nu)q}\right)
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq \left(\sum_{i=1}^{n} p_i A_i^{2p}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{2q}\right) \\
&\quad - \left(\sum_{i=1}^{n} p_i A_i^{2(1-\nu)p+2\nu q}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{2\nu p+2(1-\nu)q}\right) \\
&\quad \leq L\left(1, \left(\frac{\Gamma}{\gamma}\right)^2\right) \ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \left(\sum_{i=1}^{n} p_i A_i^{2p}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{2q}\right)
\end{aligned}$$

for $\nu \in [0,1]$.

In particular, for $\nu = 1/2$ we get

$$\begin{aligned}
&\left(\sum_{i=1}^{n} p_i A_i^{p+q}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{p+q}\right) \\
&\leq \left(\sum_{i=1}^{n} p_i A_i^{2p}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{2q}\right) \\
&\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \left(\sum_{i=1}^{n} p_i A_i^{p+q}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{p+q}\right)
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq \left(\sum_{i=1}^{n} p_i A_i^{2p}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{2q}\right) - \left(\sum_{i=1}^{n} p_i A_i^{p+q}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{p+q}\right) \\
&\leq L\left(1, \left(\frac{\Gamma}{\gamma}\right)^2\right) \ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \left(\sum_{i=1}^{n} p_i A_i^{2p}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{2q}\right).
\end{aligned}$$

The interested reader may also consider the examples $f(t) = \exp(\alpha t)$, $g(t) = \exp(\beta t)$ with $\alpha \neq \beta$ and $t \in \mathbb{R}$.

5. Conclusion

In this paper we obtained some significant operator inequalities for tensorial and Hadamard products of positive operators in Hilbert spaces. In particular, we derived generalizations of two scalar inequalities due to Tominaga that gave reverses of the celebrated Young's inequality between the arithmetic and geometric means. These achievements were accomplished by making use of the multivariate functional calculus introduced by H. Araki and F. Hansen.

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References

[1] T. Ando, *Concavity of certain maps on positive definite matrices and applications to Hadamard products*, Linear Algebra and its Applications, 26, 1979, 203-241. DOI: 10.1016/0024-3795(79)90179-4

[2] H. Araki and F. Hansen, *Jensen's operator inequality for functions of several variables*, Proceedings of the American Mathematical Society, 128(7), 2000, 2075–2084. DOI:10.1090/S0002-9939-00-05371-5

[3] J. S. Aujila and H. L. Vasudeva, *Inequalities involving Hadamard product and operator means*, Mathematica japonicae, 42(2), 1995, 265-272.

[4] S. S. Dragomir, *A trapezoid type tensorial norm inequality for continuous functions of selfadjoint operators in Hilbert spaces*, Istanbul Journal of Mathematics, 1(2), 2023, 48–56. DOI:10.26650/ijmath.2023.00006

[5] A. Korányi, *On some classes of analytic functions of several variables*, Transactions of the American Mathematical Society, 101(3), 1961, 520–554. 10.1090/S0002-9947-1961-0136765-6

[6] J. I. Fujii, *The Marcus-Khan theorem for Hilbert space operators*, Mathematica japonicae, 41(3), 1995, 531–535.

[7] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities*, Inequalities for Bounded Selfadjoint Operators on a Hilbert Space, Element, Zagreb, 2005.

[8] K. Kitamura and Y. Seo, *Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities*, Scientiae Mathematicae, 1(2), 1998, 237–241.

[9] W. Specht, *Zer Theorie der elementaren*, Mittel, Mathematische Zeitschrift, 74, 1960, 91–98.

[10] V. Stojiljković, *Twice differentiable Ostrowski type tensorial norm inequality for continuous functions of selfadjoint operators in Hilbert spaces*, Electronic Journal of Mathematical Analysis and Applications, 11(2), 2023, 1–15.

[11] V. Stojiljković and S. S. Dragomir, *Differentiable Ostrowski type tensorial norm inequality for continuous functions of selfadjoint operators in Hilbert spaces*, Gulf Journal of Mathematics, 15(2), 2023, 40-55. DOI: 10.56947/gjom.v15i2.1247

[12] M. Tominaga, Specht's ratio in the Young inequality, *Scientiae Mathematicae Japonicae Online*, 55, 2002, 583–588.

[13] S. Wada, *On some refinement of the Cauchy-Schwarz Inequality*, Linear Algebra and its Applications, 420, 2007, 433–440. DOI:10.1016/j.laa.2006.07.019