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Research Article

Hermite-Hadamard-Fejér Type Inequalities for h-convex Functions Involving ψ-Hilfer Operators

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Abstract

We use a new type of function called a B-function to create a new version of fractional Hermite-Hadamard-Fejér and trapezoid-type inequalities on the right side. To achieve this objective, we utilize h-convex functions and ψ-Hilfer operators. Furthermore, we introduce novel trapezoidal-type inequalities for specific convex classes utilizing Riemann-Liouville operators through particular instances of the principal results.

Keywords: B-function, h-convex function, incomplete beta function, ψ-Hilfer operator, Hermite-Hadamard-Fejér inequality

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1. Introduction

Convexity theory offers powerful processes and concepts to resolve a wide range of issues related to pure and applied mathematics. The use of convex functions in various mathematical fields resulted in the identification of several inequalities in the published literature. Convex integral inequalities, which are significant due to their relationship with convex functions, represent a well-established domain in real and functional analysis. These inequalities are essential for establishing limits for integrals, examining function spaces, and investigating variational difficulties. They are also fundamental to the theories of means, probability, optimization, and partial differential equations.

The Hermite-Hadamard inequality and the Hermite-Hadamard-Fejér inequality are well-known and has been used in convexity theory, see [1–6].

In [7], the author presented a new class of functions called h-convex functions.

Definition 1.1. Let $h:\mathbb{J}\subseteq\mathbb{R}\to\mathbb{R}$ where $[0,1]\subseteq\mathbb{J}$ be a positive function. We define a function $f:\mathbb{I}\to\mathbb{R}$ as h-convex if for any $x,y\in\mathbb{I}$ and $\lambda\in[0,1]$, the following condition holds:

$$f(\lambda x+(1-\lambda)y)\le h(\lambda)f(x)+h(1-\lambda)f(y). \tag{1.1}$$

In the case when the inequality (1.1) is reversed, we say that f is h-concave.

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- Definition 1.1 simplifies to the classical concept of convexity [8] by setting $h(\lambda)=\lambda$.
- By establishing $h(\lambda)=1$, Definition 1.1 is reduced to P-functions [9, 10].
- By defining $h(\lambda)=\lambda^s$, Definition 1.1 simplifies to the concept of s-convex functions in the second sens [11].

Numerous applications of fractional calculus across a wide range of disciplines contribute to its significance in mathematics. It is particularly significant in the examination of various fractional differential equations and inequalities. Recently, numerous researchers have employed a variety of mathematical methodologies to investigate the generalization of integral inequalities. One of the most popular and effective methods is the application of fractional integral operators, which convert classical integral inequalities into fractional integral inequalities. The theory addresses Riemann-Liouville fractional integrals, which constitute an essential category of fractional integral functions. Commonly referred to as ψ-Helffer operators, these represent additional fractional integral operators.

2. ψ-Hilfer operators

Definition 2.1. Let $[a,b]\subseteq[0,+\infty)$. Let $\alpha>0$ and ψ be a positive, increasing differentiable function such that $\psi'(\tau)\neq 0$ for all $\tau\in[a,b]$. The left and right sided ψ-Hilfer fractional integral of a function f with respect to the function ψ on $[a,b]$ are defined respectively as follows.

$$^{\psi}\mathfrak{J}^{\alpha}_{a^+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(\psi(x)-\psi(t))^{\alpha-1}\psi^{'}(t)f(t)dt,\quad a<x\leq b, \tag{2.1}$$

$$^{\psi}\mathfrak{J}^{\alpha}_{b^-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(\psi(t)-\psi(x))^{\alpha-1}\psi^{'}(t)f(t)dt,\quad a\leq x< b, \tag{2.2}$$

where $\Gamma(.)$ denotes the Gamma function given by

$$\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}dt,\quad \alpha\Gamma(\alpha)=\Gamma(\alpha+1).$$

For these operators, consider the following space

$$X[a,b]=\left\{f:\parallel f\parallel_X=\left(\int_a^b|f(x)|\psi'(x)dx\right)<\infty\right\}.$$

One essential property of ψ-Hilfer operators is that they are dependent on the function ψ and produce a particular type of fractional integrals.

1. Taking $\psi(\tau)=\tau$, we get Riemann-Liouville fractional operator of order $\alpha>0$

$$\mathscr{RL}^{\alpha}_{a^+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-s)^{\alpha-1}f(s)ds,\quad x>a,$$

$$\mathscr{RL}^{\alpha}_{b^-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(s-x)^{\alpha-1}f(s)ds,,\quad x<b.$$

2. Using $\psi(\tau)=\ln\tau$, we deduce Hadamard fractional operator of order $\alpha>0$

$$\mathscr{H}^{\alpha}_{a^+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\left(\ln\frac{x}{s}\right)^{\alpha-1}f(s)\frac{ds}{s},\quad x>a>1,$$

$$\mathscr{H}^{\alpha}_{a^+}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\left(\ln\frac{s}{x}\right)^{\alpha-1}f(s)\frac{ds}{s},\quad 1<x<b.$$

3. Putting $\psi(\tau)=\frac{\tau^{\rho}}{\rho}$ where $\rho>0$, we obtain Katugompola fractional operators of order $\alpha>0$.

$$\mathscr{K}_{a^+}^{\alpha} f(x)=\frac{(\rho)^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \left(x^{\rho}-s^{\rho}\right)^{\alpha-1} f(s)s^{\rho-1}ds,\ x>a,$$

$$\mathscr{K}_{b^-}^{\alpha} f(x)=\frac{(\rho)^{1-\alpha}}{\Gamma(\alpha)}\int_x^b \left(s^{\rho}-x^{\rho}\right)^{\alpha-1} f(s)s^{\rho-1}ds,\ x<b.$$

3. B-function

In recent work [12, 13], the authors introduced a novel class of functions termed B-functions, defined as follows:

Definition 3.1. Let $G:[0,\infty)\to\mathbb{R}$ be a non-negative function. The function G is called a B-function if

$$G(x-a)+G(b-x)\leq 2G\left(\frac{a+b}{2}\right), \tag{3.1}$$

where $a<x<b$ with $a,b\in[0,\infty)$.
The fact that if G is a B-function on $[a,b]$, then we denote by $G\in B(a,b)$.
If the inequality (3.1) is reversed, G is called A-function, or that G belongs to the class $A(a,b)$.
If we have equality in (3.1), G is called AB-function, or that G belongs to the class $AB(a,b)$.

Proposition 3.2. *Let $h:[0,1]\to\mathbb{R}$ be a positive function. The function h is a B-function, if for all $\lambda\in[0,1]$, we have*

$$h(\lambda)+h(1-\lambda)\leq 2h\left(\frac{1}{2}\right). \tag{3.2}$$

- *The functions $h(\lambda)=\lambda$ and $h(\lambda)=1$ are AB-function, B-function and A-function.*
- *The function $h(\lambda)=\lambda^s$, $s\in(0,1]$ is B-function.*

Proof. 1. The first case is obvious.

2. Putting $h(t)=t^s$ for $s\in(0,1]$ in the inequality (3.2) yields

$$t^s+(1-t)^s\leq 2\left(\frac{1}{2}\right)^s.$$

To prove the above result, we need the following inequality:
for all $A,B>0$, $\eta>0$, we have

$$\min(1,2^{1-\eta})(A+B)^{\eta}\leq A^{\eta}+B^{\eta}\leq \max(1,2^{1-\eta})(A+B)^{\eta}. \tag{3.3}$$

Setting $0<\eta\leq 1$ gives

$$A^{\eta}+B^{\eta}\leq 2^{1-\eta}(A+B)^{\eta},$$

taking $A=t$, $B=1-t$ and $\eta=s$ yields

$$t^s+(1-t)^s\leq 2^{1-s},$$

thus

$$t^s+(1-t)^s\leq 2\left(\frac{1}{2}\right)^s.$$

We deduce that $h(t)=t^s$ is a B-function.

□

The incomplete beta-Euler function $\beta(z,.,.)$ is defined for any $0 \leq z \leq 1$ with $p,q > 0$ as follows:

$$\beta(z,q,p) = \int_0^z (1-y)^{q-1} y^{p-1} dy.$$

We have

$$\beta(p,q) = \beta(q,p) = \beta(\lambda,p,q) + \beta(1-\lambda,q,p), \quad \forall \lambda \in [0,1]. \tag{3.4}$$

Example 3.3. For $\alpha > 0$, we have

$$\beta(1,\alpha+1) = \frac{1}{\alpha+1} \text{ and } \beta\left(\frac{1}{2},1,\alpha+1\right) = \frac{1}{(\alpha+1)2^{\alpha+1}}. \tag{3.5}$$

$$\beta(2,\alpha+1) = \frac{1}{\alpha+1} - \frac{1}{\alpha+2} \text{ and } \beta\left(\frac{1}{2},2,\alpha+1\right) = \frac{1}{(\alpha+1)2^{\alpha+1}} - \frac{1}{(\alpha+2)2^{\alpha+2}}. \tag{3.6}$$

Based on previous studies, we use ψ-Hilfer operators to create a novel version of the trapezoid inequalities and Hermite-Hadamard-Fejér for h-convex functions with h in the class $B(0,1)$.

4. Hermite-Hadamard-Fejér inequalities

We now present the first result of the Hermite-Hadamard-Fejér inequality for h-convexity of the function f involving ψ-Hilfer operators.

Theorem 4.1. *Let $\alpha > 0$, h be a B-function. Let $f \in X[a,b]$ be a h-convex function and ψ be a positive, increasing differentiable function such that $\psi'(t) \neq 0$ for all $t \in [a,b]$. If $g : [a,b] \to \mathbb{R}$ is non-negative, integrable and symmetric to $(a+b)/2$, then the following inequalities hold*

$$\begin{aligned} &\frac{1}{2h\left(\frac{1}{2}\right)} \left[{}^{\psi}\mathfrak{J}^{\alpha}_{a^+} g(b) + {}^{\psi}\mathfrak{J}^{\alpha}_{b^-} g(a)\right] f\left(\frac{a+b}{2}\right) \\ &\leq \tfrac{1}{2}\left[{}^{\psi}\mathfrak{J}^{\alpha}_{b^-} gF(a) + {}^{\psi}\mathfrak{J}^{\alpha}_{a^+} gF(b)\right] \\ &\leq 2h\left(\tfrac{1}{2}\right)\left(\frac{f(b)+f(a)}{2}\right)\left[{}^{\psi}\mathfrak{J}^{\alpha}_{a^+} g(b) + {}^{\psi}\mathfrak{J}^{\alpha}_{b^-} g(a)\right], \end{aligned} \tag{4.1}$$

where

$$F(t) = f(t) + f(a+b-t), \tag{4.2}$$

and

$$g(t) = g(a+b-t).$$

Proof. For any $t \in [a,b]$, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{1}{2}(a+b-t) + \frac{1}{2}t\right) \\ &\leq h\left(\tfrac{1}{2}\right) f(a+b-t) + h\left(\tfrac{1}{2}\right) f(t), \end{aligned}$$

then

$$f\left(\frac{a+b}{2}\right) \leq h\left(\tfrac{1}{2}\right) F(t). \tag{4.3}$$

Put $t=(1-\lambda)a+\lambda b$ in (4.2) for $\lambda\in(0,1)$ and using the h-convexity of f, we get

$$\begin{aligned} F(t) &= f((1-\lambda)b+\lambda a)+f((1-\lambda)a+\lambda b) \\ &\leq h(1-\lambda)[f(b)+f(a)]+h(\lambda)[f(b)+f(a)] \\ &= (h(\lambda)+h(1-\lambda))[f(b)+f(a)], \end{aligned}$$

given that h is a B-function, we conclude

$$F(t)\leq 2h\left(\frac{1}{2}\right)(f(b)+f(a)). \tag{4.4}$$

From (4.3) and (4.4), we deduce

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\leq\frac{1}{2}F(t)\leq 2h\left(\frac{1}{2}\right)\left(\frac{f(b)+f(a)}{2}\right). \tag{4.5}$$

Multiplying (4.5) by $(\psi(b)-\psi(t))^{\alpha-1}\psi'(t)g(t)$ and integrating over $t\in[a,b]$, we result

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right){}^{\psi}\mathfrak{J}^{\alpha}_{a^+}g(b)\leq\frac{1}{2}{}^{\psi}\mathfrak{J}^{\alpha}_{a^+}gF(b)\leq 2h\left(\frac{1}{2}\right)\left(\frac{f(b)+f(a)}{2}\right){}^{\psi}\mathfrak{J}^{\alpha}_{a^+}g(b). \tag{4.6}$$

Multiplying (4.5) by $(\psi(t)-\psi(a))^{\alpha-1}\psi'(t)g(t)$ and integrating over $t\in[a,b]$, we get

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right){}^{\psi}\mathfrak{J}^{\alpha}_{b^-}g(a)\leq\frac{1}{2}{}^{\psi}\mathfrak{J}^{\alpha}_{b^-}gF(a)\leq 2h\left(\frac{1}{2}\right)\left(\frac{f(b)+f(a)}{2}\right){}^{\psi}\mathfrak{J}^{\alpha}_{b^-}g(a). \tag{4.7}$$

Combining the inequality (4.6) with the inequality (4.7) yields the desired outcome. □

By selecting $\psi(x)=x$ and $\alpha=1$, we derive the following new formulation of the Hermite-Hadamard-Fejér inequality applicable to h-convex functions.

Corollary 4.2. *Let h be a B-function. Let $f\in L[a,b]$ be an h-convex function. If $g:[a,b]\to\mathbb{R}$ is non-negative, integrable and symmetric to $(a+b)/2$, then the following inequalities hold:*

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt\leq\int_a^b f(t)g(t)dt\leq[f(b)+f(a)]\,h\left(\frac{1}{2}\right)\int_a^b g(t)dt, \tag{4.8}$$

where

$$g(t)=g(a+b-t).$$

Remark 4.3. Taking $h(t)=t$ in (4.8), we get the Hermite-Hadamard-Fejér inequality for convex function in [14].

The following results are dependent on the function h presented in Theorem 4.1.

Corollary 4.4. *Let $\alpha>0$, $f\in X[a,b]$, ψ be a positive, increasing differentiable function such that $\psi'(t)\neq 0$ for all $t\in[a,b]$ and $F(t)=f(\tau)+f(a+b-\tau)$. Let $g:[a,b]\to\mathbb{R}$ be a non-negative, integrable function and symmetric to $(a+b)/2$ (i. e $g(t)=g(a+b-t)$.*

1. *Taking $h(\lambda)=\lambda$ gives f as a convex function, then the following inequalities hold:*

$$\begin{aligned} \left[{}^{\psi}\mathfrak{J}^{\alpha}_{a^+}g(b)+{}^{\psi}\mathfrak{J}^{\alpha}_{b^-}g(a)\right]f\left(\frac{a+b}{2}\right) &\leq\tfrac{1}{2}\left[{}^{\psi}\mathfrak{J}^{\alpha}_{b^-}gF(a)+{}^{\psi}\mathfrak{J}^{\alpha}_{a^+}gF(b)\right] \\ &\leq\left(\frac{f(b)+f(a)}{2}\right)\left[{}^{\psi}\mathfrak{J}^{\alpha}_{a^+}g(b)+{}^{\psi}\mathfrak{J}^{\alpha}_{b^-}g(a)\right]. \end{aligned} \tag{4.9}$$

The result is for the convex functions defined in [15].

2. *Setting $h(\lambda)=1$ results in f being a P-functions, then the following inequalities hold:*

$$\begin{aligned}\frac{1}{2}\left[{}^{\psi}\mathfrak{J}^{\alpha}_{a^+}g(b)+{}^{\psi}\mathfrak{J}^{\alpha}_{b^-}g(a)\right]f\left(\frac{a+b}{2}\right) &\leq \tfrac{1}{2}\left[{}^{\psi}\mathfrak{J}^{\alpha}_{b^-}gF(a)+{}^{\psi}\mathfrak{J}^{\alpha}_{a^+}gF(b)\right]\\ &\leq 2\left(\frac{f(b)+f(a)}{2}\right)\left[{}^{\psi}\mathfrak{J}^{\alpha}_{a^+}g(b)+{}^{\psi}\mathfrak{J}^{\alpha}_{b^-}g(a)\right].\end{aligned} \tag{4.10}$$

3. *Putting $h(\lambda)=\lambda^s$ with $s\in(0,1]$ results in f becoming a s-convex function therefore, the following inequalities are valid:*

$$\begin{aligned}2^{s-1}\left[{}^{\psi}\mathfrak{J}^{\alpha}_{a^+}g(b)+{}^{\psi}\mathfrak{J}^{\alpha}_{b^-}g(a)\right]f\left(\frac{a+b}{2}\right) &\leq \tfrac{1}{2}\left[{}^{\psi}\mathfrak{J}^{\alpha}_{b^-}gF(a)+{}^{\psi}\mathfrak{J}^{\alpha}_{a^+}gF(b)\right]\\ &\leq \left(\frac{f(b)+f(a)}{2^s}\right)\left[{}^{\psi}\mathfrak{J}^{\alpha}_{a^+}g(b)+{}^{\psi}\mathfrak{J}^{\alpha}_{b^-}g(a)\right].\end{aligned} \tag{4.11}$$

Remark 4.5. Using $\psi(\tau)=\tau$, $\psi(\tau)=\ln\tau$, and $\psi(\tau)=\frac{\tau^\rho}{\rho}$ in Corollary 4.4, we obtain Hermite-Hadamard-Fejér inequality for convex, P-function and s-convex functions involving Riemann-Liouville fractional operators, Hadamard fractional operators and Katugompola fractional operators, respectively.

5. Trapezoid type inequality

The second result requires the following Lemma: [15, Lemma 2].

Lemma 5.1. *If α,ψ and g are defined as in theorem 4.1 and $f:[a,b]\to\mathbb{R}$ is a differentiable mapping to (a,b), then the following identity holds:*

$$\begin{aligned}&\left(\frac{f(b)+f(a)}{2}\right)\left[{}^{\psi}\mathfrak{J}^{\alpha}_{a^+}g(b)+{}^{\psi}\mathfrak{J}^{\alpha}_{b^-}g(a)\right]-\tfrac{1}{2}\left[{}^{\psi}\mathfrak{J}^{\alpha}_{b^-}gF(a)+{}^{\psi}\mathfrak{J}^{\alpha}_{a^+}gF(b)\right]\\ &=\frac{1}{2\Gamma(\alpha)}\int_a^b\Lambda_{\psi,g}(\tau)f'(\tau)d\tau,\end{aligned} \tag{5.1}$$

where

$$\Lambda_{\psi,g}(\tau)=\int\limits_{a+b-\tau}^{\tau}\frac{\psi'(r)g(r)}{(\psi(r)-\psi(a))^{1-\alpha}}dr+\int\limits_{a+b-\tau}^{\tau}\frac{\psi'(r)g(r)}{(\psi(b)-\psi(r))^{1-\alpha}}dr. \tag{5.2}$$

Remark 5.2. The function $\Lambda_{\psi,g}(\tau)$ is non-decreasing on $[a,b]$, and verify the following:

$$\begin{cases}\Lambda_{\psi,g}(\tau)\leq 0 & if\ a\leq\tau\leq\frac{a+b}{2},\\ \Lambda_{\psi,g}(\tau)>0 & if\ \frac{a+b}{2}<\tau\leq b.\end{cases} \tag{5.3}$$

For all $x,\,y\in(a,\,b)$, we define the operator $\Phi^{\psi}_{g,h}(x,y)$ as:

$$\begin{aligned}\Phi^{\psi}_{g,h}(x,y) &= \int\limits_a^{\frac{a+b}{2}}\left(\int\limits_{\tau}^{a+b-\tau}\frac{\psi'(r)g(r)}{|\psi(y)-\psi(r)|^{1-\alpha}}dr\right)h\left(\frac{|x-\tau|}{b-a}\right)d\tau\\ &+\int\limits_{\frac{a+b}{2}}^{b}\left(\int\limits_{a+b-\tau}^{\tau}\frac{\psi'(r)g(r)}{|\psi(y)-\psi(r)|^{1-\alpha}}dr\right)h\left(\frac{|x-\tau|}{b-a}\right)d\tau.\end{aligned} \tag{5.4}$$

The operator $\Phi_{g,h}^{\psi}$ has the following basic properties:

$$\begin{aligned}\Phi_{g,h}^{\psi}(b,a) &= \int_{a}^{\frac{a+b}{2}}\left(\int_{\tau}^{a+b-\tau}\frac{\psi'(r)g(r)}{[\psi(r)-\psi(a)]^{1-\alpha}}dr\right)h\left(\frac{b-\tau}{b-a}\right)d\tau \\ &+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-\tau}^{\tau}\frac{\psi'(r)g(r)}{[\psi(r)-\psi(a)]^{1-\alpha}}dr\right)h\left(\frac{b-\tau}{b-a}\right)d\tau \\ &=\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\frac{\psi'(r)g(r)}{[\psi(r)-\psi(a)]^{1-\alpha}}dr\right)h\left(\frac{t-a}{b-a}\right)dt \\ &+\int_{a}^{\frac{a+b}{2}}\left(\int_{t}^{a+b-t}\frac{\psi'(r)g(r)}{[\psi(r)-\psi(a)]^{1-\alpha}}dr\right)h\left(\frac{t-a}{b-a}\right)dt \\ &=\Phi_{g,h}^{\psi}(a,a)\end{aligned}$$

similarly,

$$\Phi_{g,h}^{\psi}(a,b)=\Phi_{g,h}^{\psi}(b,b).$$

Consequently, we have the following equalities

$$\Phi_{g,h}^{\psi}(b,a)=\Phi_{g,h}^{\psi}(a,a) \text{ and } \Phi_{g,h}^{\psi}(a,b)=\Phi_{g,h}^{\psi}(b,b). \tag{5.5}$$

Theorem 5.3. *Assume h is a B-function and α,ψ and g are defined according to Theorem 4.1. If $|f'|$ is a h-convex mapping on $[a,b]$, then the trapezoid type inequality is obtained as:*

$$\begin{aligned}&\left|\left(\frac{f(b)+f(a)}{2}\right)\left[{}^{\psi}\mathfrak{J}_{a^+}^{\alpha}g(b)+{}^{\psi}\mathfrak{J}_{b^-}^{\alpha}g(a)\right]-\tfrac{1}{2}\left[{}^{\psi}\mathfrak{J}_{b^-}^{\alpha}gF(a)+{}^{\psi}\mathfrak{J}_{a^+}^{\alpha}gF(b)\right]\right| \\ &\le\frac{\Phi_{g,h}^{\psi}(a,a)+\Phi_{g,h}^{\psi}(b,b)}{2\Gamma(\alpha)}\Big[|f'(a)|+|f'(b)|\Big].\end{aligned} \tag{5.6}$$

Proof. Using the absolute value of identity (5.1) and the h-convexity of the function $|f'|$, we deduce

$$|f'(\tau)|=\left|f'\left(\frac{b-\tau}{b-a}a+\frac{\tau-a}{b-a}b\right)\right|\le h\left(\frac{b-\tau}{b-a}\right)\left|f'(a)\right|+h\left(\frac{\tau-a}{b-a}\right)\left|f'(b)\right|,$$

then

$$\begin{aligned}&\left|\left(\frac{f(b)+f(a)}{2}\right)\left[{}^{\psi}\mathfrak{J}_{a^+}^{\alpha}g(b)+{}^{\psi}\mathfrak{J}_{b^-}^{\alpha}g(a)\right]-\tfrac{1}{2}\left[{}^{\psi}\mathfrak{J}_{b^-}^{\alpha}gF(a)+{}^{\psi}\mathfrak{J}_{a^+}^{\alpha}gF(b)\right]\right| \\ &\le\frac{1}{2\Gamma(\alpha)}\int_a^b\left|\Lambda_{\psi,g}(\tau)\right|\left|f'(\tau)\right|d\tau \\ &\le\frac{|f'(a)|}{2\Gamma(\alpha)}\int_a^b\left|\Lambda_{\psi,g}(\tau)\right|h\left(\frac{b-\tau}{b-a}\right)d\tau+\frac{|f'(b)|}{2\Gamma(\alpha)}\int_a^b\left|\Lambda_{\psi,g}(\tau)\right|h\left(\frac{\tau-a}{b-a}\right)d\tau.\end{aligned} \tag{5.7}$$

From the equalities (5.2) and (5.4), we derive

$$\begin{aligned}
&\int_a^b \left|\Lambda_{\psi,g}(\tau)\right| h\left(\frac{b-\tau}{b-a}\right) d\tau \\
&= \int_a^{\frac{a+b}{2}} \left(\int_{\tau}^{a+b-\tau} \frac{\psi'(r)g(r)}{(\psi(r)-\psi(a))^{1-\alpha}} dr \right) h\left(\frac{b-\tau}{b-a}\right) d\tau \\
&+ \int_a^{\frac{a+b}{2}} \left(\int_{\tau}^{a+b-\tau} \frac{\psi'(r)g(r)}{(\psi(b)-\psi(r))^{1-\alpha}} dr \right) h\left(\frac{b-\tau}{b-a}\right) d\tau \\
&+ \int_{\frac{a+b}{2}}^{b} \left(\int_{a+b-\tau}^{\tau} \frac{\psi'(r)g(r)}{(\psi(r)-\psi(a))^{1-\alpha}} dr \right) h\left(\frac{b-\tau}{b-a}\right) d\tau \\
&+ \int_{\frac{a+b}{2}}^{b} \left(\int_{a+b-\tau}^{\tau} \frac{\psi'(r)g(r)}{(\psi(b)-\psi(r))^{1-\alpha}} dr \right) h\left(\frac{b-\tau}{b-a}\right) d\tau \\
&= \Phi_{g,h}^{\psi}(b,a) + \Phi_{g,h}^{\psi}(b,b).
\end{aligned}$$

Thus

$$\int_a^b \left|\Lambda_{\psi,g}(\tau)\right| h\left(\frac{b-\tau}{b-a}\right) d\tau = \Phi_{g,h}^{\psi}(b,a) + \Phi_{g,h}^{\psi}(b,b). \tag{5.8}$$

Also, it is observable that

$$\int_a^b \left|\Lambda_{\psi,g}(\tau)\right| h\left(\frac{\tau-a}{b-a}\right) d\tau = \Phi_{g,h}^{\psi}(a,a) + \Phi_{g,h}^{\psi}(a,b). \tag{5.9}$$

The desired result is obtained by replacing the equalities (5.8) and (5.9) into (5.7). □

Using $h(x) = x$, we derive the following Corollary, see ([15, Theorem.7]).

Corollary 5.4. *Assume α, ψ and g are defined according to Theorem 4.1. If $|f'|$ is a convex mapping on $[a,b]$, then the trapezoid type inequality is obtained as:*

$$\begin{aligned}
&\left| \left(\frac{f(b)+f(a)}{2}\right) \left[{}^{\psi}\mathfrak{I}_{a^+}^{\alpha} g(b) + {}^{\psi}\mathfrak{I}_{b^-}^{\alpha} g(a)\right] - \tfrac{1}{2}\left[{}^{\psi}\mathfrak{I}_{b^-}^{\alpha} gF(a) + {}^{\psi}\mathfrak{I}_{a^+}^{\alpha} gF(b)\right] \right| \\
&\leq \frac{H_g^{\psi}(a,a) + H_g^{\psi}(b,b)}{2(b-a)\Gamma(\alpha)} \Big[|f'(a)| + |f'(b)|\Big],
\end{aligned} \tag{5.10}$$

where

$$\begin{aligned}
H_g^{\psi}(x,y) &= \int_a^{\frac{a+b}{2}} \left(\int_{\tau}^{a+b-\tau} \frac{\psi'(r)g(r)}{|\psi(y)-\psi(r)|^{1-\alpha}} dr \right) |x-\tau| d\tau \\
&+ \int_{\frac{a+b}{2}}^{b} \left(\int_{a+b-\tau}^{\tau} \frac{\psi'(r)g(r)}{|\psi(y)-\psi(r)|^{1-\alpha}} dr \right) |x-\tau| d\tau.
\end{aligned}$$

When evaluating $\psi(x) = x$ and $\|g\|_{\infty} = \sup_{r\in(a,b)} |g(r)|$, consider the following cases:

1. Given $h(\lambda)=\lambda^s$ with $s\in(0,1]$, we obtain

$$\Phi^{\psi}_{g,h}(a,a)\le\frac{\|g\|_\infty}{(b-a)^s}$$
$$\times\left\{\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}(r-a)^{\alpha-1}dr\right)(t-a)^s dt+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}(r-a)^{\alpha-1}dr\right)(t-a)^s dt\right\}. \tag{5.11}$$

Taking $t=b-(\frac{b-a}{2})x$, we get

$$\begin{aligned}\int_{\frac{a+b}{2}}^{b}(b-t)^\alpha(t-a)^s dt &=\int_0^1\left(\frac{b-a}{2}x\right)^\alpha\left(b-a-\frac{b-a}{2}x\right)^s\left(\frac{b-a}{2}\right)dx\\ &=(b-a)^{\alpha+s+1}\int_0^1\left(\frac{x}{2}\right)^\alpha\left(1-\frac{x}{2}\right)^s d\left(\frac{x}{2}\right)\\ &=(b-a)^{\alpha+s+1}\int_0^{\frac12}(1-y)^s y^\alpha dy\\ &=(b-a)^{\alpha+s+1}\beta\left(\tfrac12,s+1,\alpha+1\right).\end{aligned} \tag{5.12}$$

For $t=a+\left(\frac{b-a}{2}\right)x$, we get

$$\begin{aligned}\int_a^{\frac{a+b}{2}}(b-t)^\alpha(t-a)^s dt &=\int_0^1\left(b-a-\left(\frac{b-a}{2}\right)x\right)^\alpha\left(\left(\frac{b-a}{2}\right)x\right)^s\left(\frac{b-a}{2}\right)dx\\ &=(b-a)^{\alpha+s+1}\int_0^1\left(1-\frac{x}{2}\right)^\alpha\left(\frac{x}{2}\right)^s d\left(\frac{x}{2}\right)\\ &=(b-a)^{\alpha+s+1}\beta\left(\tfrac12,\alpha+1,s+1\right).\end{aligned}$$

On the other side, we have

$$\begin{aligned}&\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}(r-a)^{\alpha-1}dr\right)(t-a)^s dt\\ &=\frac{1}{\alpha}\left[\int_a^{\frac{a+b}{2}}(b-t)^\alpha(t-a)^s dt-\int_a^{\frac{a+b}{2}}(t-a)^{\alpha+s}dt\right]\\ &=\frac{1}{\alpha}\left[\int_a^{\frac{a+b}{2}}(b-t)^\alpha(t-a)^s dt-\frac{1}{\alpha+s+1}\left(\frac{b-a}{2}\right)^{\alpha+s+1}\right]\\ &=\frac{(b-a)^{\alpha+s+1}}{\alpha}\left[\beta\left(\frac12,\alpha+1,s+1\right)-\frac{1}{(\alpha+s+1)2^{\alpha+s+1}}\right],\end{aligned} \tag{5.13}$$

and

$$\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}(r-a)^{\alpha-1}dr\right)(t-a)^{s}dt$$
$$=\frac{1}{\alpha}\left[\int_{\frac{a+b}{2}}^{b}(t-a)^{\alpha+s}dt-\int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha}(t-a)^{s}dt\right] \tag{5.14}$$
$$=\frac{(b-a)^{\alpha+s+1}}{\alpha}\left[\frac{1}{\alpha+s+1}-\frac{1}{(\alpha+s+1)2^{\alpha+s+1}}-\beta\left(\frac{1}{2},s+1,\alpha+1\right)\right].$$

Based on (5.11), (3.4), (5.13), and (5.14), we conclude

$$\Phi_{g,h}^{\psi}(a,a)\leq\frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{\alpha}\times\left[\frac{1}{\alpha+s+1}\left(1-\frac{1}{2^{\alpha+s}}\right)+\beta\left(s+1,\alpha+1\right)-2\beta\left(\frac{1}{2},s+1,\alpha+1\right)\right].$$

Analogously, we get

$$\Phi_{g,h}^{\psi}(b,b)\leq\frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{\alpha}\times\left[\frac{1}{\alpha+s+1}\left(1-\frac{1}{2^{\alpha+s}}\right)+\beta\left(s+1,\alpha+1\right)-2\beta\left(\frac{1}{2},s+1,\alpha+1\right)\right].$$

A new result is obtained by applying the previous inequalities.

Corollary 5.5. *Assume α and g are defined according to Theorem 4.1. If $|f'|$ is a s-convex function on $[a,b]$, then the trapezoid type inequality is obtained as*

$$\left|\left(\frac{f(b)+f(a)}{2}\right)\left[\mathfrak{J}_{a^+}^{\alpha}g(b)+\mathfrak{J}_{b^-}^{\alpha}g(a)\right]-\frac{1}{2}\left[\mathfrak{J}_{b^-}^{\alpha}gF(a)+\mathfrak{J}_{a^+}^{\alpha}gF(b)\right]\right|\leq\frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{\Gamma(\alpha+1)}$$
$$\times\left[\frac{1}{\alpha+s+1}\left(1-\frac{1}{2^{\alpha+s}}\right)+\beta\left(s+1,\alpha+1\right)-2\beta\left(\frac{1}{2},s+1,\alpha+1\right)\right]\left[|f'(a)|+|f'(b)|\right]. \tag{5.15}$$

2. Setting $s=1$ yields $h(\lambda)=\lambda$, and by applying (3.6), we have the subsequent corollary.

Corollary 5.6. *Assume α and g are defined according to Theorem 4.1. If $|f'|$ is a convex mapping on $[a,b]$, then the trapezoid type inequality is obtained as*

$$\left|\left(\frac{f(b)+f(a)}{2}\right)\left[\mathfrak{J}_{a^+}^{\alpha}g(b)+\mathfrak{J}_{b^-}^{\alpha}g(a)\right]-\frac{1}{2}\left[\mathfrak{J}_{b^-}^{\alpha}gF(a)+\mathfrak{J}_{a^+}^{\alpha}gF(b)\right]\right|$$
$$\leq\frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{\Gamma(\alpha+2)}\left(1-\frac{1}{2^{\alpha}}\right)\left[|f'(a)|+|f'(b)|\right]. \tag{5.16}$$

The result is the same as the [16, Theorem.2.6].

3. Setting $s\to0^{+}$ obtains $h(\lambda)=1$. Applying (3.5) yields the following new result related to the class P-function.

Corollary 5.7. *Assume α and g are defined according to Theorem 4.1. If $|f'|$ is a P-function on $[a,b]$, then the trapezoid type inequality is obtained as*

$$\begin{aligned} &\left| \left(\frac{f(b)+f(a)}{2} \right) \left[\mathfrak{J}^{\alpha}_{a^+} g(b) + \mathfrak{J}^{\alpha}_{b^-} g(a) \right] - \tfrac{1}{2} \left[\mathfrak{J}^{\alpha}_{b^-} gF(a) + \mathfrak{J}^{\alpha}_{a^+} gF(b) \right] \right| \\ &\leq \frac{2(b-a)^{\alpha+1} \|g\|_\infty}{\Gamma(\alpha+2)} \left(1 - \frac{1}{2^\alpha} \right) \left[|f'(a)| + |f'(b)| \right]. \end{aligned} \tag{5.17}$$

6. Conclusion

A lot of authors are still working on different fractional operators to study the integral inequalities that make a lot of different inequalities more general. In fractional calculus, one thing that is similar is that the ψ-Hilfer operator is used to repeat some integral conditions for certain function classes. In this article, we come up with some Hermite-Hadamard-Fejér-type inequalities for h-convex functions that use ψ-Hilfer operators. By looking at this study, we also found that different types of convexity were linked to results that had already been published. In a future work, researchers will show similar inequality for linked convex functions and other types of functions.

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