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Research Article

A Study of Some New Ostrowski's Type Integral Inequalities via Multi-step Linear Kernel

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Abstract

In recent years, significant progress has been made in the study of integral inequalities, with particular emphasis on Ostrowski-type inequalities. These inequalities have found wide-ranging applications in various mathematical and scientific domains, including numerical quadrature, statistics, probability theory, transform theory, and the estimation of special functions. In this paper, we present new results under different norms by employing a novel multi-step linear kernel. Additionally, several interesting and noteworthy findings are established.

Keywords: Ostrowski inequality, numerical integration, multi-step linear kernel

2020 MSC: 26D15, 26A33, 26D07, 26D10.

1. Introduction

The Ostrowski inequality [1] has found applications across a wide range of disciplines, including data science, machine learning, and coding theory. It also plays a significant role in other scientific domains such as geophysics, bioengineering, biology [2, 3], and demography [4, 5]. Numerous researchers have studied Ostrowski-type inequalities and their diverse applications (see, for example, [6, 8, 11, 13]). Historically, variants of Peano kernels have been employed in various contexts to derive meaningful inequalities.

This paper focuses on a broad class of modifications to the Ostrowski inequality using a specialized 14-step Peano kernel. By leveraging tools from modern inequality theory, we establish explicit and rigorous bounds [7, 9, 10, 12, 14]. The analysis incorporates foundational inequalities such as the Grüss inequality, the Diaz–Metcalf inequality, and the Cauchy inequality to derive new and significant results. Several error estimates and insightful remarks are also provided (see, for example, [15, 16, 19]).

Using these foundational inequalities [17, 18], namely the Grüss, Diaz–Metcalf, and Cauchy inequalities, we derive results under various smoothness conditions, including $\phi' \in L^1$, $\phi' \in L^2$, and $\phi'' \in L^2$. Recent mathematical research has placed considerable emphasis on constructing new Ostrowski-type inequalities using multistep linear

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kernels. These inequalities are particularly useful for integral estimation and have significant applications in numerical integration and functional analysis.

Several contemporary studies have proposed generalized forms of Ostrowski-type inequalities using linear kernels, with a particular focus on multistep techniques. One such study introduces a weighted Ostrowski-type inequality based on a novel three-step kernel. Building upon earlier findings and incorporating the Grüss inequality, this study establishes new results with implications for mathematical analysis and numerical approximation.

Furthermore, recent developments in related inequality frameworks, such as Bullen-type inequalities for twice-differentiable mappings [20], operator-based inequalities in Hilbert spaces [21], and Hermite-Hadamard-type inequalities involving ψ-conformable fractional integrals [22], have further enriched the theoretical landscape. These advances offer new pathways for extending Ostrowski-type results and unifying them with broader classes of functional and operator inequalities.

2. Main Results

To proceed with our main findings related to the 14-step linear kernel, we first establish the following lemma.

Lemma 2.1. *Let $\mathscr{L} : [\breve{u}, \mathfrak{b}] \to \mathbb{R}$ be a function such that $\mathscr{L}'$ is absolutely continuous on $[\breve{u}, \mathfrak{b}]$. Define the kernel $p(\chi, \mathfrak{k})$ as:*

$$p(\chi,\mathfrak{k}) = \begin{cases} \mathfrak{k}-\breve{u}, & \mathfrak{k}\in\left(\breve{u}, \dfrac{63\breve{u}+\chi}{64}\right] \\ \mathfrak{k}-\dfrac{127\breve{u}+\mathfrak{b}}{128}, & \mathfrak{k}\in\left(\dfrac{63\breve{u}+\chi}{64}, \dfrac{31\breve{u}+\chi}{32}\right] \\ \mathfrak{k}-\dfrac{63\breve{u}+\mathfrak{b}}{64}, & \mathfrak{k}\in\left(\dfrac{31\breve{u}+\chi}{32}, \dfrac{15\breve{u}+\chi}{16}\right] \\ \mathfrak{k}-\dfrac{31\breve{u}+\mathfrak{b}}{32}, & \mathfrak{k}\in\left(\dfrac{15\breve{u}+\chi}{16}, \dfrac{7\breve{u}+\chi}{8}\right] \\ \mathfrak{k}-\dfrac{15\breve{u}+\mathfrak{b}}{16}, & \mathfrak{k}\in\left(\dfrac{7\breve{u}+\chi}{8}, \dfrac{3\breve{u}+\chi}{4}\right] \\ \mathfrak{k}-\dfrac{7\breve{u}+\mathfrak{b}}{8}, & \mathfrak{k}\in\left(\dfrac{3\breve{u}+\chi}{4}, \dfrac{\breve{u}+\chi}{2}\right] \\ \mathfrak{k}-\dfrac{3\breve{u}+\mathfrak{b}}{4}, & \mathfrak{k}\in\left(\dfrac{\breve{u}+\chi}{2}, \chi\right] \\ \mathfrak{k}-\dfrac{\breve{u}+\mathfrak{b}}{2}, & \mathfrak{k}\in\left(\chi, \breve{u}+\mathfrak{b}-\chi\right] \\ \mathfrak{k}-\dfrac{\breve{u}+3\mathfrak{b}}{4}, & \mathfrak{k}\in\left(\breve{u}+\mathfrak{b}-\chi, \dfrac{\breve{u}+2\mathfrak{b}-\chi}{2}\right] \\ \mathfrak{k}-\dfrac{\breve{u}+7\mathfrak{b}}{8}, & \mathfrak{k}\in\left(\dfrac{\breve{u}+2\mathfrak{b}-\chi}{2}, \dfrac{\breve{u}+4\mathfrak{b}-\chi}{4}\right] \\ \mathfrak{k}-\dfrac{\breve{u}+15\mathfrak{b}}{16}, & \mathfrak{k}\in\left(\dfrac{\breve{u}+4\mathfrak{b}-\chi}{4}, \dfrac{\breve{u}+8\mathfrak{b}-\chi}{8}\right] \\ \mathfrak{k}-\dfrac{\breve{u}+31\mathfrak{b}}{32}, & \mathfrak{k}\in\left(\dfrac{\breve{u}+8\mathfrak{b}-\chi}{8}, \dfrac{\breve{u}+16\mathfrak{b}-\chi}{16}\right] \\ \mathfrak{k}-\dfrac{\breve{u}+63\mathfrak{b}}{64}, & \mathfrak{k}\in\left(\dfrac{\breve{u}+16\mathfrak{b}-\chi}{16}, \dfrac{\breve{u}+32\mathfrak{b}-\chi}{32}\right] \\ \mathfrak{k}-\mathfrak{b}, & \mathfrak{k}\in\left(\dfrac{\breve{u}+32\mathfrak{b}-\chi}{32}, \mathfrak{b}\right] \end{cases} \tag{2.1}$$

for $\forall \chi \in \left[\breve{u}, \dfrac{\breve{u}+\mathfrak{b}}{2}\right]$, then we have the following identity:

$$\frac{1}{\mathfrak{b}-\breve{u}} \int_{\breve{u}}^{\mathfrak{b}} p(\chi,\mathfrak{k})\, \mathscr{L}'(\mathfrak{k})\, d\mathfrak{k}$$

$$
\begin{aligned}
&= \frac{1}{4}\left[\frac{1}{32}\left\{\mathscr{L}\left(\frac{31\breve{u}+\chi}{32}\right)+\mathscr{L}\left(\frac{63\breve{u}+\chi}{64}\right)\right\}+\frac{1}{16}\left\{\mathscr{L}\left(\frac{15\breve{u}+\chi}{16}\right)\right.\right. \\
&+\mathscr{L}\left(\frac{\breve{u}+16\mathfrak{b}-\chi}{16}\right)+\mathscr{L}\left(\frac{\breve{u}+32\mathfrak{b}-\chi}{32}\right)\right\}+\frac{1}{8}\left\{\mathscr{L}\left(\frac{7\breve{u}+\chi}{8}\right)\right. \\
&+\mathscr{L}\left(\frac{\breve{u}+8\mathfrak{b}-\chi}{8}\right)\right\}+\frac{1}{4}\left\{\mathscr{L}\left(\frac{3\breve{u}+\chi}{4}\right)+\mathscr{L}\left(\frac{\breve{u}+4\mathfrak{b}-\chi}{4}\right)\right\} \\
&+\frac{1}{2}\left\{\mathscr{L}\left(\frac{\breve{u}+\chi}{2}\right)+\mathscr{L}\left(\frac{\breve{u}+2\mathfrak{b}-\chi}{2}\right)\right\}+\left\{\mathscr{L}(\chi)+\mathscr{L}(\breve{u}+\mathfrak{b}-\chi)\right\}\Big] \\
&-\frac{1}{\mathfrak{b}-\breve{u}}\int_{\breve{u}}^{\mathfrak{b}}\mathscr{L}(\mathfrak{k})\,d\mathfrak{k}.
\end{aligned}
\tag{2.2}
$$

Proof. By using (2.1), we get

$$
\begin{aligned}
&\int_{\breve{u}}^{\mathfrak{b}} p(\chi,\mathfrak{k})\mathscr{L}'(\mathfrak{k})\,d\mathfrak{k} \\
&= \int_{\breve{u}}^{\frac{63\breve{u}+\chi}{64}}(\mathfrak{k}-\breve{u})\mathscr{L}'(\mathfrak{k})\,d\mathfrak{k}+\int_{\frac{63\breve{u}+\chi}{64}}^{\frac{31\breve{u}+\chi}{32}}\left(\mathfrak{k}-\frac{127\breve{u}+\mathfrak{b}}{128}\right)\mathscr{L}'(\mathfrak{k})\,d\mathfrak{k} \\
&+\int_{\frac{31\breve{u}+\chi}{32}}^{\frac{15\breve{u}+\chi}{16}}\left(\mathfrak{k}-\frac{63\breve{u}+\mathfrak{b}}{64}\right)\mathscr{L}'(\mathfrak{k})\,d\mathfrak{k}+\int_{\frac{15\breve{u}+\chi}{16}}^{\frac{7\breve{u}+\chi}{8}}\left(\mathfrak{k}-\frac{31\breve{u}+\mathfrak{b}}{32}\right)\mathscr{L}'(\mathfrak{k})\,d\mathfrak{k} \\
&+\int_{\frac{7\breve{u}+\chi}{8}}^{\frac{3\breve{u}+\chi}{4}}\left(\mathfrak{k}-\frac{15\breve{u}+\mathfrak{b}}{16}\right)\mathscr{L}'(\mathfrak{k})\,d\mathfrak{k}+\int_{\frac{3\breve{u}+\chi}{4}}^{\frac{\breve{u}+\chi}{2}}\left(\mathfrak{k}-\frac{7\breve{u}+\mathfrak{b}}{8}\right)\mathscr{L}'(\mathfrak{k})\,d\mathfrak{k} \\
&+\int_{\frac{\breve{u}+\chi}{2}}^{\chi}\left(\mathfrak{k}-\frac{3\breve{u}+\mathfrak{b}}{4}\right)\mathscr{L}'(\mathfrak{k})\,d\mathfrak{k}+\int_{\chi}^{\breve{u}+\mathfrak{b}-\chi}\left(\mathfrak{k}-\frac{\breve{u}+\mathfrak{b}}{2}\right)\mathscr{L}'(\mathfrak{k})\,d\mathfrak{k} \\
&+\int_{\breve{u}+\mathfrak{b}-\chi}^{\frac{\breve{u}+2\mathfrak{b}-\chi}{2}}\left(\mathfrak{k}-\frac{\breve{u}+3\mathfrak{b}}{4}\right)\mathscr{L}'(\mathfrak{k})\,d\mathfrak{k}+\int_{\frac{\breve{u}+2\mathfrak{b}-\chi}{2}}^{\frac{\breve{u}+4\mathfrak{b}-\chi}{4}}\left(\mathfrak{k}-\frac{\breve{u}+7\mathfrak{b}}{8}\right)\mathscr{L}'(\mathfrak{k})\,d\mathfrak{k} \\
&+\int_{\frac{\breve{u}+4\mathfrak{b}-\chi}{4}}^{\frac{\breve{u}+8\mathfrak{b}-\chi}{8}}\left(\mathfrak{k}-\frac{\breve{u}+15\mathfrak{b}}{16}\right)\mathscr{L}'(\mathfrak{k})\,d\mathfrak{k}+\int_{\frac{\breve{u}+8\mathfrak{b}-\chi}{8}}^{\frac{\breve{u}+16\mathfrak{b}-\chi}{16}}\left(\mathfrak{k}-\frac{\breve{u}+31\mathfrak{b}}{32}\right)\mathscr{L}'(\mathfrak{k})\,d\mathfrak{k} \\
&+\int_{\frac{\breve{u}+16\mathfrak{b}-\chi}{16}}^{\frac{\breve{u}+32\mathfrak{b}-\chi}{32}}\left(\mathfrak{k}-\frac{\breve{u}+63\mathfrak{b}}{64}\right)\mathscr{L}'(\mathfrak{k})\,d\mathfrak{k}+\int_{\frac{\breve{u}+32\mathfrak{b}-\chi}{32}}^{\mathfrak{b}}(\mathfrak{k}-\mathfrak{b})\mathscr{L}'(\mathfrak{k})\,d\mathfrak{k}.
\end{aligned}
\tag{2.3}
$$

By applying integration by parts, we get the required identity (2.2).

□

Now, we proceed to present three distinct cases based on the above identity.

Case 1(a) For $\mathscr{L} \in L_1[\hat{u}, \check{b}]$

Theorem 2.2. *Let* $\mathscr{L}:[\breve{u},\mathfrak{b}]\to\mathbb{R}$ *be differentiable on* $(\breve{u},\mathfrak{b})$. *If* $\mathscr{L}'\in L^1[\breve{u},\mathfrak{b}]$*is absolutely continuous on* $[\breve{u},\mathfrak{b}]$ *and* $\gamma\le\mathscr{L}'(\mathfrak{k})\le s, \forall\, \mathfrak{k}\in[\breve{u},\mathfrak{b}]$, *then the following inequality holds*

$$
\begin{aligned}
&\left|\frac{1}{4}\left[\frac{1}{32}\left\{\mathscr{L}\left(\frac{31\breve{u}+\chi}{32}\right)+\mathscr{L}\left(\frac{63\breve{u}+\chi}{64}\right)\right\}+\frac{1}{16}\left\{\mathscr{L}\left(\frac{31\breve{u}+\chi}{32}\right)\right.\right.\right. \\
&+\mathscr{L}\left(\frac{63\breve{u}+\chi}{64}\right)\right\}+\frac{1}{8}\left\{\mathscr{L}\left(\frac{7\breve{u}+\chi}{8}\right)+\mathscr{L}\left(\frac{\breve{u}+8\mathfrak{b}-\chi}{8}\right)\right\}
\end{aligned}
\tag{2.4}
$$

$$+\frac{1}{4}\left\{\mathscr{L}\left(\frac{3\breve{u}+\chi}{4}\right)+\mathscr{L}\left(\frac{\breve{u}+4\mathfrak{b}-\chi}{4}\right)\right\}+\frac{1}{2}\left\{\mathscr{L}\left(\frac{\breve{u}+\chi}{2}\right)\right.$$
$$\left.+\mathscr{L}\left(\frac{\breve{u}+2\mathfrak{b}-\chi}{2}\right)\right\}+\left\{\mathscr{L}(\chi)+\mathscr{L}(\breve{u}+\mathfrak{b}-\chi)\right\}\Big]-\frac{1}{\mathfrak{b}-\breve{u}}\int_{\breve{u}}^{\mathfrak{b}}\mathscr{L}(\mathfrak{k})\,d\mathfrak{k}$$
$$-\frac{\mathscr{L}(\mathfrak{b})-\mathscr{L}(\breve{u})}{(\mathfrak{b}-\breve{u})^2}\left\{\frac{1}{2048}\left\{\frac{-3}{4}(\chi-\breve{u})^2+\left(\chi-\frac{3\breve{u}+\mathfrak{b}}{4}\right)^2\right.\right.$$
$$\left.\left.-\frac{1}{4}\left(\chi-\frac{\breve{u}+\mathfrak{b}}{2}\right)^2\right\}\right\}\Bigg|$$
$$\leq \upsilon(\chi)(\mathfrak{b}-\breve{u})(s-\gamma),$$

where

$$\upsilon(\chi)=\max\left\{(\chi-\breve{u})^2,\left(\chi-\frac{\breve{u}+\mathfrak{b}}{2}\right)^2,\left(\chi-\frac{3\breve{u}+\mathfrak{b}}{4}\right)^2\right\}.$$

Proof. As we know that for all $\mathfrak{k}\in[\breve{u},\mathfrak{b}]$ and $\chi\in\left[\breve{u},\frac{\breve{u}+\mathfrak{b}}{2}\right]$, we have

$$\chi-\frac{127\breve{u}+\mathfrak{b}}{128}\leq p(\chi,\mathfrak{k})\leq \chi-\breve{u}.$$

Apply Grüss-inequality to the mappings $P(\chi,\mathfrak{k})$ and $\mathscr{L}'(\mathfrak{k})$, we obtain

$$\left|\frac{1}{\mathfrak{b}-\breve{u}}\int_{\breve{u}}^{\mathfrak{b}}p(\chi,\mathfrak{k})\mathscr{L}'(\mathfrak{k})\,d\mathfrak{k}-\frac{1}{(\mathfrak{b}-\breve{u})^2}\int_{\breve{u}}^{\mathfrak{b}}p(\chi,\mathfrak{k})\int_{\breve{u}}^{\mathfrak{b}}\mathscr{L}'(\mathfrak{k})\,d\mathfrak{k}\right| \tag{2.5}$$
$$\leq \upsilon(\chi)(\mathfrak{b}-\breve{u})(s-\gamma).$$

Now,

$$\frac{1}{\mathfrak{b}-\breve{u}}\int_{\breve{u}}^{\mathfrak{b}}p(\chi,\mathfrak{k})\,d\mathfrak{k} \tag{2.6}$$
$$=\frac{1}{\mathfrak{b}-\breve{u}}\left[\int_{\breve{u}}^{\frac{63\breve{u}+\chi}{64}}(\mathfrak{k}-\breve{u})\,d\mathfrak{k}+\int_{\frac{63\breve{u}+\chi}{64}}^{\frac{31\breve{u}+\chi}{32}}\left(\mathfrak{k}-\frac{127\breve{u}+\mathfrak{b}}{128}\right)d\mathfrak{k}\right.$$
$$+\int_{\frac{31\breve{u}+\chi}{32}}^{\frac{15\breve{u}+\chi}{16}}\left(\mathfrak{k}-\frac{63\breve{u}+\mathfrak{b}}{64}\right)d\mathfrak{k}+\int_{\frac{15\breve{u}+\chi}{16}}^{\frac{7\breve{u}+\chi}{8}}\left(\mathfrak{k}-\frac{31\breve{u}+\mathfrak{b}}{32}\right)d\mathfrak{k}$$
$$+\int_{\frac{7\breve{u}+\chi}{8}}^{\frac{3\breve{u}+\chi}{4}}\left(\mathfrak{k}-\frac{15\breve{u}+\mathfrak{b}}{16}\right)d\mathfrak{k}+\int_{\frac{3\breve{u}+\chi}{4}}^{\frac{\breve{u}+\chi}{2}}\left(\mathfrak{k}-\frac{7\breve{u}+\mathfrak{b}}{8}\right)d\mathfrak{k}$$
$$+\int_{\frac{\breve{u}+\chi}{2}}^{\chi}\left(\mathfrak{k}-\frac{3\breve{u}+\mathfrak{b}}{4}\right)d\mathfrak{k}+\int_{\chi}^{\breve{u}+\mathfrak{b}-\chi}\left(\mathfrak{k}-\frac{\breve{u}+\mathfrak{b}}{2}\right)d\mathfrak{k}$$
$$+\int_{\breve{u}+\mathfrak{b}-\chi}^{\frac{\breve{u}+2\mathfrak{b}-\chi}{2}}\left(\mathfrak{k}-\frac{\breve{u}+3\mathfrak{b}}{4}\right)d\mathfrak{k}+\int_{\frac{\breve{u}+2\mathfrak{b}-\chi}{2}}^{\frac{\breve{u}+4\mathfrak{b}-\chi}{4}}\left(\mathfrak{k}-\frac{\breve{u}+7\mathfrak{b}}{8}\right)d\mathfrak{k}$$
$$+\int_{\frac{\breve{u}+4\mathfrak{b}-\chi}{4}}^{\frac{\breve{u}+8\mathfrak{b}-\chi}{8}}\left(\mathfrak{k}-\frac{\breve{u}+15\mathfrak{b}}{16}\right)d\mathfrak{k}+\int_{\frac{\breve{u}+8\mathfrak{b}-\chi}{8}}^{\frac{\breve{u}+16\mathfrak{b}-\chi}{16}}\left(\mathfrak{k}-\frac{\breve{u}+31\mathfrak{b}}{32}\right)d\mathfrak{k}$$
$$\left.+\int_{\frac{\breve{u}+16\mathfrak{b}-\chi}{16}}^{\frac{\breve{u}+32\mathfrak{b}-\chi}{32}}\left(\mathfrak{k}-\frac{\breve{u}+63\mathfrak{b}}{64}\right)d\mathfrak{k}+\int_{\frac{\breve{u}+32\mathfrak{b}-\chi}{32}}^{\mathfrak{b}}(\mathfrak{k}-\mathfrak{b})\,d\mathfrak{k}\right].$$

□

Corollary 2.3. *By replacing* $\chi = \frac{\breve{u}+\mathfrak{b}}{2}$ *in* (2.4)*, then we have*

$$\left|\frac{1}{4}\left[\frac{1}{32}\left\{\mathscr{L}\left(\frac{63\breve{u}+\mathfrak{b}}{64}\right)+\mathscr{L}\left(\frac{127\breve{u}+\mathfrak{b}}{128}\right)\right\}\right.\right. \tag{2.7}$$
$$+\frac{1}{16}\left\{\mathscr{L}\left(\frac{31\breve{u}+\mathfrak{b}}{32}\right)+\mathscr{L}\left(\frac{\breve{u}+31\mathfrak{b}}{32}\right)+\mathscr{L}\left(\frac{\breve{u}+63\mathfrak{b}}{64}\right)\right\}$$
$$+\frac{1}{8}\left\{\mathscr{L}\left(\frac{15\breve{u}+\mathfrak{b}}{16}\right)+\mathscr{L}\left(\frac{\breve{u}+15\mathfrak{b}}{16}\right)\right\}$$
$$+\frac{1}{4}\left\{\mathscr{L}\left(\frac{7\breve{u}+\mathfrak{b}}{8}\right)+\mathscr{L}\left(\frac{\breve{u}+7\mathfrak{b}}{8}\right)\right\}$$
$$+\frac{1}{2}\left\{\mathscr{L}\left(\frac{3\breve{u}+\mathfrak{b}}{4}\right)+\mathscr{L}\left(\frac{\breve{u}+3\mathfrak{b}}{4}\right)\right\}$$
$$\left.\left.+2\mathscr{L}\left(\frac{\breve{u}+\mathfrak{b}}{2}\right)\right]-\frac{1}{\mathfrak{b}-\breve{u}}\int_{\breve{u}}^{\mathfrak{b}}\mathscr{L}(\mathfrak{k})\,d\mathfrak{k}+\frac{\{\mathscr{L}(\mathfrak{b})-\mathscr{L}(\breve{u})\}}{16384}\right|$$
$$\leq \upsilon\left(\frac{\breve{u}+\mathfrak{b}}{2}\right)(\mathfrak{b}-\breve{u})(s-\gamma).$$

Case 1(b) For $\mathscr{L}' \in L_1[\breve{u},\mathfrak{b}]$

Theorem 2.4. *Let* $I :\subset \mathbb{R} \to \mathbb{R}$ *be differentiable mapping on* I^0*, the interior of the interval* I*, and let* $\breve{u},\mathfrak{b} \in I$ *with* $\breve{u} < \mathfrak{b}$*. If* $\mathscr{L}' \in L^1[\breve{u},\mathfrak{b}]$ *and* $\gamma \leq \mathscr{L}'(\mathfrak{k}) \leq \Gamma, \forall\, \chi \in [\breve{u},\mathfrak{b}]$*. Then the following inequality holds*

$$\left|\frac{1}{4}\left[\frac{1}{32}\left\{\mathscr{L}\left(\frac{31\breve{u}+\chi}{32}\right)+\mathscr{L}\left(\frac{63\breve{u}+\chi}{64}\right)\right\}\right.\right. \tag{2.8}$$
$$+\frac{1}{16}\left\{\mathscr{L}\left(\frac{15\breve{u}+\chi}{16}\right)+\mathscr{L}\left(\frac{\breve{u}+16\mathfrak{b}-\chi}{16}\right)+\mathscr{L}\left(\frac{\breve{u}+32\mathfrak{b}-\chi}{32}\right)\right\}$$
$$+\frac{1}{8}\left\{\mathscr{L}\left(\frac{7\breve{u}+\chi}{8}\right)+\mathscr{L}\left(\frac{\breve{u}+8\mathfrak{b}-\chi}{8}\right)\right\}$$
$$+\frac{1}{4}\left\{\mathscr{L}\left(\frac{3\breve{u}+\chi}{4}\right)+\mathscr{L}\left(\frac{\breve{u}+4\mathfrak{b}-\chi}{4}\right)\right\}$$
$$+\frac{1}{2}\left\{\mathscr{L}\left(\frac{\breve{u}+\chi}{2}\right)+\mathscr{L}\left(\frac{\breve{u}+2\mathfrak{b}-\chi}{2}\right)\right\}$$
$$\left.+\{\mathscr{L}(\chi)+\mathscr{L}(\breve{u}+\mathfrak{b}-\chi)\}\right]-\frac{1}{\mathfrak{b}-\breve{u}}\int_{\breve{u}}^{\mathfrak{b}}\mathscr{L}(\mathfrak{k})\,d\mathfrak{k}$$
$$\left.-\frac{1}{8192(\mathfrak{b}-\breve{u})}\left\{-3(\chi-\breve{u})^2+4\left(\chi-\frac{3\breve{u}+\mathfrak{b}}{4}\right)^2-\left(\chi-\frac{\breve{u}+\mathfrak{b}}{2}\right)^2\right\}\right|$$
$$\leq \frac{\Gamma-\gamma}{16384(\mathfrak{b}-\breve{u})}\left\{-3(\chi-\breve{u})^2+4\left(\chi-\frac{3\breve{u}+\mathfrak{b}}{4}\right)^2-\left(\chi-\frac{\breve{u}+\mathfrak{b}}{2}\right)^2\right\},$$

$\forall \chi \in \left[\breve{u}, \frac{\breve{u}+\mathfrak{b}}{2}\right]$.

Corollary 2.5. *By replacing* $\chi = \frac{\breve{u}+\mathfrak{b}}{2}$ *in* (2.8), *then we have*

$$\left|\frac{1}{4}\left[\frac{1}{32}\left\{\mathscr{L}\left(\frac{63\breve{u}+\mathfrak{b}}{64}\right)+\mathscr{L}\left(\frac{127\breve{u}+\mathfrak{b}}{128}\right)\right\}\right.\right.$$
$$+\frac{1}{16}\left\{\mathscr{L}\left(\frac{31\breve{u}+\mathfrak{b}}{32}\right)+\mathscr{L}\left(\frac{\breve{u}+31\mathfrak{b}}{32}\right)+\mathscr{L}\left(\frac{\breve{u}+63\mathfrak{b}}{64}\right)\right\}$$
$$+\frac{1}{8}\left\{\mathscr{L}\left(\frac{15\breve{u}+\mathfrak{b}}{16}\right)+\mathscr{L}\left(\frac{\breve{u}+15\mathfrak{b}}{16}\right)\right\}$$
$$+\frac{1}{4}\left\{\mathscr{L}\left(\frac{7\breve{u}+\mathfrak{b}}{8}\right)+\mathscr{L}\left(\frac{\breve{u}+7\mathfrak{b}}{8}\right)\right\}$$
$$+\frac{1}{2}\left\{\mathscr{L}\left(\frac{3\breve{u}+\mathfrak{b}}{4}\right)+\mathscr{L}\left(\frac{\breve{u}+3\mathfrak{b}}{4}\right)\right\}$$
$$\left.\left.+2\mathscr{L}\left(\frac{\breve{u}+\mathfrak{b}}{2}\right)\right]-\frac{1}{\mathfrak{b}-\breve{u}}\int_{\breve{u}}^{\mathfrak{b}}\mathscr{L}(\mathfrak{k})\,d\mathfrak{k}+\frac{\mathfrak{b}-\breve{u}}{16384}\right|$$
$$\leq \frac{-1}{32768}(\mathfrak{b}-\breve{u})(\Gamma-\gamma).$$

Case 1(c) for $\mathscr{L} \in L_1[\hat{u}, \check{b}]$.

Theorem 2.6. *Let* $\mathscr{L} : [\breve{u}, \mathfrak{b}] \to \mathbb{R}$ *be differentiable mapping on* $(\breve{u}, \mathfrak{b})$. *If* $\mathscr{L}' \in L^1[\breve{u}, \mathfrak{b}]$ *and* $\gamma \leq \mathscr{L}'(\mathfrak{k}) \leq \Gamma, \forall\, \mathfrak{k} \in [\breve{u}, \mathfrak{b}]$. *Then the following inequality holds*

$$\left|\frac{1}{4}\left[\frac{1}{32}\left\{\mathscr{L}\left(\frac{31\breve{u}+\chi}{32}\right)+\mathscr{L}\left(\frac{63\breve{u}+\chi}{64}\right)\right\}\right.\right. \tag{2.9}$$
$$+\frac{1}{16}\left\{\mathscr{L}\left(\frac{15\breve{u}+\chi}{16}\right)+\mathscr{L}\left(\frac{\breve{u}+16\mathfrak{b}-\chi}{16}\right)+\mathscr{L}\left(\frac{\breve{u}+32\mathfrak{b}-\chi}{32}\right)\right\}$$
$$+\frac{1}{8}\left\{\mathscr{L}\left(\frac{7\breve{u}+\chi}{8}\right)+\mathscr{L}\left(\frac{\breve{u}+8\mathfrak{b}-\chi}{8}\right)\right\}$$
$$+\frac{1}{4}\left\{\mathscr{L}\left(\frac{3\breve{u}+\chi}{4}\right)+\mathscr{L}\left(\frac{\breve{u}+4\mathfrak{b}-\chi}{4}\right)\right\}$$
$$+\frac{1}{2}\left\{\mathscr{L}\left(\frac{\breve{u}+\chi}{2}\right)+\mathscr{L}\left(\frac{\breve{u}+2\mathfrak{b}-\chi}{2}\right)\right\}$$
$$+\left.\{\mathscr{L}(\chi)+\mathscr{L}(\breve{u}+\mathfrak{b}-\chi)\}\right]-\frac{1}{\mathfrak{b}-\breve{u}}\int_{\breve{u}}^{\mathfrak{b}}\mathscr{L}(\mathfrak{k})\,d\mathfrak{k}$$
$$\left.-\frac{1}{8192(\mathfrak{b}-\breve{u})}\left\{-3(\chi-\breve{u})^2+4\left(\chi-\frac{3\breve{u}+\mathfrak{b}}{4}\right)^2-\left(\chi-\frac{\breve{u}+\mathfrak{b}}{2}\right)^2\right\}\right|$$
$$\leq \Omega(S-\gamma)$$

and

$$\left|\frac{1}{4}\left[\frac{1}{32}\left\{\mathscr{L}\left(\frac{31\breve{u}+\chi}{32}\right)+\mathscr{L}\left(\frac{63\breve{u}+\chi}{64}\right)\right\}\right.\right. \tag{2.10}$$
$$+\frac{1}{16}\left\{\mathscr{L}\left(\frac{15\breve{u}+\chi}{16}\right)+\mathscr{L}\left(\frac{\breve{u}+16\mathfrak{b}-\chi}{16}\right)+\mathscr{L}\left(\frac{\breve{u}+32\mathfrak{b}-\chi}{32}\right)\right\}$$

$$
\begin{aligned}
&+\frac{1}{8}\left\{\mathscr{L}\left(\frac{7\breve{u}+\chi}{8}\right)+\mathscr{L}\left(\frac{\breve{u}+8\mathfrak{b}-\chi}{8}\right)\right\}\\
&+\frac{1}{4}\left\{\mathscr{L}\left(\frac{3\breve{u}+\chi}{4}\right)+\mathscr{L}\left(\frac{\breve{u}+4\mathfrak{b}-\chi}{4}\right)\right\}\\
&+\frac{1}{2}\left\{\mathscr{L}\left(\frac{\breve{u}+\chi}{2}\right)+\mathscr{L}\left(\frac{\breve{u}+2\mathfrak{b}-\chi}{2}\right)\right\}\\
&+\left\{\mathscr{L}(\chi)+\mathscr{L}(\breve{u}+\mathfrak{b}-\chi)\right\}\Big]-\frac{1}{\mathfrak{b}-\breve{u}}\int_{\breve{u}}^{\mathfrak{b}}\mathscr{L}(\mathfrak{k})\,d\mathfrak{k}\\
&-\frac{1}{8192(\mathfrak{b}-\breve{u})}\left\{-3(\chi-\breve{u})^2+4\left(\chi-\frac{3\breve{u}+\mathfrak{b}}{4}\right)^2-\left(\chi-\frac{\breve{u}+\mathfrak{b}}{2}\right)^2\right\}\Bigg|\\
&\leq \Omega(S-\gamma),
\end{aligned}
$$

$\forall \chi \in \left[\breve{u}, \frac{\breve{u}+\mathfrak{b}}{2}\right]$, *where*

$$
\begin{aligned}
\Omega &= \max_{\mathfrak{k}\in[\breve{u},\mathfrak{b}]}|p(\chi,\mathfrak{k})|,\\
S &= \frac{\mathscr{L}(\mathfrak{b})-\mathscr{L}(\breve{u})}{\mathfrak{b}-\breve{u}},\\
\gamma &= \inf_{\mathfrak{k}\in[\breve{u},\mathfrak{b}]}\mathscr{L}'(\mathfrak{k}),\\
\Gamma &= \sup_{\mathfrak{k}\in[\breve{u},\mathfrak{b}]}\mathscr{L}'(\mathfrak{k}).
\end{aligned}
$$

Case 2 For $\mathscr{L}' \in L^2[\breve{u},\mathfrak{b}]$

Theorem 2.7. *Let* $\mathscr{L}:[\breve{u},\mathfrak{b}]\to\mathbb{R}$ *be three times differentiable function on* $(\breve{u},\mathfrak{b})$. *If* $\mathscr{L}'\in L^2[\breve{u},\mathfrak{b}]$, *then the following inequality holds*

$$
\begin{aligned}
&\Bigg|\frac{1}{4}\Bigg[\frac{1}{32}\left\{\mathscr{L}\left(\frac{31\breve{u}+\chi}{32}\right)+\mathscr{L}\left(\frac{63\breve{u}+\chi}{64}\right)\right\}+\frac{1}{16}\left\{\mathscr{L}\left(\frac{15\breve{u}+\chi}{16}\right)\right.\\
&\left.+\mathscr{L}\left(\frac{\breve{u}+16\mathfrak{b}-\chi}{16}\right)+\mathscr{L}\left(\frac{\breve{u}+32\mathfrak{b}-\chi}{32}\right)\right\}+\frac{1}{8}\left\{\mathscr{L}\left(\frac{7\breve{u}+\chi}{8}\right)\right.\\
&\left.+\mathscr{L}\left(\frac{\breve{u}+8\mathfrak{b}-\chi}{8}\right)\right\}+\frac{1}{4}\left\{\mathscr{L}\left(\frac{3\breve{u}+\chi}{4}\right)+\mathscr{L}\left(\frac{\breve{u}+4\mathfrak{b}-\chi}{4}\right)\right\}\\
&+\frac{1}{2}\left\{\mathscr{L}\left(\frac{\breve{u}+\chi}{2}\right)+\mathscr{L}\left(\frac{\breve{u}+2\mathfrak{b}-\chi}{2}\right)\right\}+\mathscr{L}(\chi)+\mathscr{L}(\breve{u}+\mathfrak{b}-\chi)\Bigg]\\
&-\frac{1}{\mathfrak{b}-\breve{u}}\int_{\breve{u}}^{\mathfrak{b}}\mathscr{L}(\mathfrak{k})\,d\mathfrak{k}-\frac{\mathscr{L}(\mathfrak{b})-\mathscr{L}(\breve{u})}{(\mathfrak{b}-\breve{u})^2}\Bigg[\frac{1}{2048}\left\{-\frac{3}{4}(\chi-\breve{u})^2\right.\\
&\left.+\left(\chi-\frac{3\breve{u}+\mathfrak{b}}{4}\right)^2-\frac{1}{4}\left(\chi-\frac{\breve{u}+\mathfrak{b}}{2}\right)^2\right\}\Bigg]\Bigg|\\
&\leq\Bigg[\frac{9}{786432}(\chi-\breve{u})^3+\frac{74897}{98304}\left(\chi-\frac{3\breve{u}+\mathfrak{b}}{4}\right)^3-\frac{599185}{786432}\left(\chi-\frac{\breve{u}+\mathfrak{b}}{2}\right)^3
\end{aligned}
\tag{2.11}
$$

$$-\frac{1}{\mathfrak{b}-\check{u}}\left[\frac{1}{2048}\left\{-\frac{3}{4}(\chi-\check{u})^2+\left(\chi-\frac{3\check{u}+\mathfrak{b}}{4}\right)^2\right.\right.$$
$$\left.\left.-\frac{1}{4}\left(\chi-\frac{\check{u}+\mathfrak{b}}{2}\right)^2\right\}\right]^2\Bigg]^{\frac{1}{2}}\frac{\sqrt{\sigma(\mathscr{L}')}}{\sqrt{\mathfrak{b}-\check{u}}},$$

$\forall\chi\in\left[\check{u},\frac{\check{u}+\mathfrak{b}}{2}\right]$, *where*

$$\sigma(\mathscr{L}')=\left\|\mathscr{L}'\right\|^2-\frac{(\mathscr{L}(\mathfrak{b})-\mathscr{L}(\check{u}))^2}{\mathfrak{b}-\check{u}}$$
$$=\left\|\mathscr{L}'\right\|_2^2-s^2(\mathfrak{b}-\check{u})$$
$$s=\frac{\mathscr{L}(\mathfrak{b})-\mathscr{L}(\check{u})}{\mathfrak{b}-\check{u}}.$$

Corollary 2.8. *By replacing* $\chi=\frac{\check{u}+\mathfrak{b}}{2}$ *in* (2.11)*, then we have*

$$\left|\frac{1}{4}\left[\frac{1}{32}\left\{\mathscr{L}\left(\frac{63\check{u}+\mathfrak{b}}{64}\right)+\mathscr{L}\left(\frac{127\check{u}+\mathfrak{b}}{128}\right)\right\}\right.\right. \tag{2.12}$$
$$+\frac{1}{16}\left\{\mathscr{L}\left(\frac{31\check{u}+\mathfrak{b}}{32}\right)+\mathscr{L}\left(\frac{\check{u}+31\mathfrak{b}}{32}\right)+\mathscr{L}\left(\frac{\check{u}+63\mathfrak{b}}{64}\right)\right\}$$
$$+\frac{1}{8}\left\{\mathscr{L}\left(\frac{15\check{u}+\mathfrak{b}}{16}\right)+\mathscr{L}\left(\frac{\check{u}+15\mathfrak{b}}{16}\right)\right\}$$
$$+\frac{1}{4}\left\{\mathscr{L}\left(\frac{7\check{u}+\mathfrak{b}}{8}\right)+\mathscr{L}\left(\frac{\check{u}+7\mathfrak{b}}{8}\right)\right\}$$
$$+\frac{1}{2}\left\{\mathscr{L}\left(\frac{3\check{u}+\mathfrak{b}}{4}\right)+\mathscr{L}\left(\frac{\check{u}+3\mathfrak{b}}{4}\right)\right\}$$
$$\left.\left.+2\mathscr{L}\left(\frac{\check{u}+\mathfrak{b}}{2}\right)\right]-\frac{1}{\mathfrak{b}-\check{u}}\int_{\check{u}}^{\mathfrak{b}}\mathscr{L}(\mathfrak{k})\,d\mathfrak{k}+\frac{\mathscr{L}(\mathfrak{b})-\mathscr{L}(\check{u})}{16384}\right|$$
$$\leq\frac{1}{16384}\sqrt{\frac{9575165}{3}}\sqrt{\sigma(\mathscr{L}')}(\mathfrak{b}-\check{u}).$$

Case 3 For $\mathscr{L}''\in L^2[\check{u},\mathfrak{b}]$

Theorem 2.9. *Let* $\mathscr{L}:[\check{u},\mathfrak{b}]\to\mathbb{R}$ *be a twice absolutely continous differentiable mapping on* $(\check{u},\mathfrak{b})$, *with* $\mathscr{L}''\in L^2[\check{u},\mathfrak{b}]$. *Then the following inequality holds*

$$\left|\frac{1}{4}\left[\frac{1}{32}\left\{\mathscr{L}\left(\frac{31\check{u}+\chi}{32}\right)+\mathscr{L}\left(\frac{63\check{u}+\chi}{64}\right)\right\}+\frac{1}{16}\left\{\mathscr{L}\left(\frac{15\check{u}+\chi}{16}\right)\right.\right.\right. \tag{2.13}$$
$$\left.+\mathscr{L}\left(\frac{\check{u}+16\mathfrak{b}-\chi}{16}\right)+\mathscr{L}\left(\frac{\check{u}+32\mathfrak{b}-\chi}{32}\right)\right\}+\frac{1}{8}\left\{\mathscr{L}\left(\frac{7\check{u}+\chi}{8}\right)\right.$$
$$\left.+\mathscr{L}\left(\frac{\check{u}+8\mathfrak{b}-\chi}{8}\right)\right\}+\frac{1}{4}\left\{\mathscr{L}\left(\frac{3\check{u}+\chi}{4}\right)+\mathscr{L}\left(\frac{\check{u}+4\mathfrak{b}-\chi}{4}\right)\right\}$$
$$+\frac{1}{2}\left\{\mathscr{L}\left(\frac{\check{u}+\chi}{2}\right)+\mathscr{L}\left(\frac{\check{u}+2\mathfrak{b}-\chi}{2}\right)\right\}+\mathscr{L}(\chi)+\mathscr{L}(\check{u}+\mathfrak{b}-\chi)\Big]$$

$$-\frac{1}{\mathfrak{b}-\breve{u}}\int_{\breve{u}}^{\mathfrak{b}}\mathscr{L}(\mathfrak{k})\,d\mathfrak{k}-\frac{\mathscr{L}(\mathfrak{b})-\mathscr{L}(\breve{u})}{(\mathfrak{b}-\breve{u})^2}\left[\frac{1}{2048}\left\{-\frac{3}{4}(\chi-\breve{u})^2\right.\right.$$
$$\left.\left.\left.+\left(\chi-\frac{3\breve{u}+\mathfrak{b}}{4}\right)^2-\frac{1}{4}\left(\chi-\frac{\breve{u}+\mathfrak{b}}{2}\right)^2\right\}\right]\right|$$
$$\leq\left[\frac{9}{786432}(\chi-\breve{u})^3+\frac{74897}{98304}\left(\chi-\frac{3\breve{u}+\mathfrak{b}}{4}\right)^3-\frac{599185}{786432}\left(\chi-\frac{\breve{u}+\mathfrak{b}}{2}\right)^3\right.$$
$$-\frac{1}{\mathfrak{b}-\breve{u}}\left[\frac{1}{2048}\left\{-\frac{3}{4}(\chi-\breve{u})^2+\left(\chi-\frac{3\breve{u}+\mathfrak{b}}{4}\right)^2\right.\right.$$
$$\left.\left.\left.-\frac{1}{4}\left(\chi-\frac{\breve{u}+\mathfrak{b}}{2}\right)^2\right\}\right]^2\right]^{\frac{1}{2}}\frac{1}{\pi}\left\|\mathscr{L}''\right\|_2,$$

$\forall\,\chi\in\left[\breve{u},\frac{\breve{u}+\mathfrak{b}}{2}\right]$.

Corollary 2.10. *By replacing $\chi=\frac{\breve{u}+\mathfrak{b}}{2}$ in (2.12), then we have*

$$\left|\frac{1}{4}\left[\frac{1}{32}\left\{\mathscr{L}\left(\frac{63\breve{u}+\mathfrak{b}}{64}\right)+\mathscr{L}\left(\frac{127\breve{u}+\mathfrak{b}}{128}\right)\right\}\right.\right. \tag{2.14}$$
$$+\frac{1}{16}\left\{\mathscr{L}\left(\frac{31\breve{u}+\mathfrak{b}}{32}\right)+\mathscr{L}\left(\frac{\breve{u}+31\mathfrak{b}}{32}\right)+\mathscr{L}\left(\frac{\breve{u}+63\mathfrak{b}}{64}\right)\right\}$$
$$+\frac{1}{8}\left\{\mathscr{L}\left(\frac{15\breve{u}+\mathfrak{b}}{16}\right)+\mathscr{L}\left(\frac{\breve{u}+15\mathfrak{b}}{16}\right)\right\}$$
$$+\frac{1}{4}\left\{\mathscr{L}\left(\frac{7\breve{u}+\mathfrak{b}}{8}\right)+\mathscr{L}\left(\frac{\breve{u}+7\mathfrak{b}}{8}\right)\right\}$$
$$+\frac{1}{2}\left\{\mathscr{L}\left(\frac{3\breve{u}+\mathfrak{b}}{4}\right)+\mathscr{L}\left(\frac{\breve{u}+3\mathfrak{b}}{4}\right)\right\}$$
$$\left.\left.+2\mathscr{L}\left(\frac{\breve{u}+\mathfrak{b}}{2}\right)\right]-\frac{1}{\mathfrak{b}-\breve{u}}\int_{\breve{u}}^{\mathfrak{b}}\mathscr{L}(\mathfrak{k})\,d\mathfrak{k}+\frac{\mathscr{L}(\mathfrak{b})-\mathscr{L}(\breve{u})}{16384}\right|$$
$$\leq\frac{1}{16384}\sqrt{\frac{9575165}{3}}\,\frac{1}{\pi}\left\|\mathscr{L}''\right\|_2(\mathfrak{b}-\breve{u})^{\frac{3}{2}}.$$

3. Conclusion

In this paper, we have developed new generalized Ostrowski-type inequalities for specific norms using a multistep linear kernel. Several related results have also been established in the form of corollaries. The presented results can be further extended to n-times differentiable functions under various norms, to functions of bounded variation, and to their corresponding weighted versions.

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