J. Inequal. Math. Anal., 1(1), 2025, 1–14

Journal of Inequalities and Mathematical Analysis

ISSN:3062-3251

Journal Homepage: www.inequalmath.com

Research Article

Theoretical Results on a Special Two-parameter Trivariate Hilbert-type Integral Inequality

Christophe Chesneau

Department of Mathematics, LMNO, University of Caen-Normandie, 14032 Caen, France.

Abstract

Hilbert integral inequalities of various types have been widely studied in mathematics. There is still room for research in this area, especially in the trivariate case, which has received less attention than the bivariate case. This article contributes to this topic by studying a special trivariate Hilbert-type integral inequality, which has the property of depending on three functions and several adjustable parameters. Four theoretical results are derived under different assumptions, the first three of which establish upper bounds, while the last one establishes a lower bound. These inequalities provide mathematical tools useful for solving three-dimensional problems involving complex integrals.

Keywords: Hilbert-type integral inequalities, Hölder integral inequality, Gamma function, Lower bounds

2020 MSC: 26D15, 33E20.

1. Introduction

This article begins with a brief review of the state of the art on Hilbert-type integral inequalities, the new contributions, and the organization of the subsequent sections.

1.1. State of the art

Several results laid the foundations of modern mathematics. The Hilbert integral inequality is one of them: it gives a sharp upper bound on the double integrals of certain ratio-product functions, which are crucial in analysis. Let us state explicitly a general version of this inequality, sometimes called the Hardy-Hilbert integral inequality. For any $p,q>1$ such that $1/p+1/q=1$ and $f,g:[0,+\infty)\to[0,+\infty)$ such that $0<\int_0^{+\infty} f^p(x)dx<+\infty$ and $0<\int_0^{+\infty} g^q(y)dy<+\infty$, the following holds:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x+y}dxdy\le \Xi\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where

$$\Xi=\frac{\pi}{\sin(\pi/p)}.$$

The technical details can be found in [11, 27]. The case $p=q=2$, corresponding to the classical Hilbert integral inequality, gives the famous constant π as the factor constant, i.e., $\Xi=\pi$. In the bivariate case, the Hilbert integral

Email address: christophe.chesneau@gmail.com (Christophe Chesneau)

doi: 10.63286/jima.2025.01

Received: 13 December 2024 Revised: 16 January 2025 Accepted: 21 January 2025

inequality has served as the basis for many generalizations and extensions established over the years. They can depend on several adjustable parameters, different kernel functions and different differential operators. A short list of important references on the subject is [22, 24, 21, 25, 15, 26, 9, 20, 4, 1, 2, 8]. In addition to these generalizations and extensions, some connections between the Hilbert integral inequality and the famous Hardy integral inequality are established in [3]. The survey in [7] also provides a comprehensive overview of the current state of the art. It summarizes recent progress and highlights open problems for future research.

Beyond the bivariate case, the study of multivariate variants of the Hilbert integral inequality has attracted attention. These variants mechanically involve additional complexity due to the interaction of more than three variables and functions. Some relevant references can be found in [14, 23, 16]. We also cite [12, 6, 29, 18, 28, 17, 13] for recent contributions to the topic.

For the purposes of this article, it is important to recall a specific trivariate integral inequality that is derived from the main result in [23]. Let us state this result explicitly. For any $p,q,r>1$ such that $1/p+1/q+1/r=1$, $\lambda>3-\min(p,q,r)$, and $f,g,h:[0,+\infty)\to[0,+\infty)$, such that $0<\int_0^{+\infty}x^{2-\lambda}f^p(x)dx<+\infty$, $0<\int_0^{+\infty}y^{2-\lambda}g^q(y)dy<+\infty$ and $0<\int_0^{+\infty}z^{2-\lambda}h^r(z)dz<+\infty$, the following holds:

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{(x+y+z)^{\lambda}}dxdydz\\&\leq \Upsilon\left[\int_0^{+\infty}x^{2-\lambda}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{2-\lambda}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}x^{2-\lambda}h^r(z)dz\right]^{1/r},\end{aligned}$$

where

$$\Upsilon=\frac{\Gamma(1+(\lambda-3)/p)\Gamma(1+(\lambda-3)/q)\Gamma(1+(\lambda-3)/r)}{\Gamma(\lambda)}$$

and $\Gamma(\alpha)=\int_0^{+\infty}t^{\alpha-1}e^{-t}dt$ with $\alpha>0$ denotes the standard gamma function taken at α. This trivariate integral inequality thus depends on three functions, f, g and h, and extends the bivariate Hilbert integral inequality by introducing an additional dimension and an adjustable parameter, λ. This addition increases the flexibility of the inequality and broadens its applicability. It also highlights the importance of the gamma function in determining the sharp factor constant Υ, which does not hold in the simpler bivariate case.

This key trivariate integral result has inspired several others, including its bounded integration interval version, which is based on the following triple integral:

$$\int_0^{a}\int_0^{a}\int_0^{a}\frac{f(x)g(y)h(z)}{(x+y+z)^{\lambda}}dxdydz$$

with $a>0$. It is discussed in detail in [16], with the exact definition of the upper bound, including the factor constant. The trivariate Hilbert-type integral inequalities studied in [17] are also remarkable for their originality and complexity. They are based on the following two triple integrals:

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{|x-y-z|^{\lambda}}dxdydz$$

and

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{|1-xyz|^{\lambda}}dxdydz.$$

Their complexity lies in the interaction of the variables x, y and z, combined with the adjustable parameter λ, which plays a role in the integrability assumptions. As shown in [17], the sharp factor constants derived for the corresponding inequalities are related to the beta function, which has a known ratio-product expression in terms of the gamma function (their exact expressions are omitted here for brevity). The proofs of the corresponding inequalities are based on three steps. First, technical decompositions of the triple integrals are considered, introducing three intermediate terms adapted to the specific situation. Second, the generalized Hölder integral inequality is used to separate these three terms in a sense (see [19, 5] for more information on the generalized Hölder integral inequality). Third, various techniques from integral calculus are used to evaluate and simplify the obtained expressions.

These results demonstrate original approaches that extend bivariate Hilbert-type integral inequalities to higher dimensions. They establish precise bounds that are critical for advancing theoretical insights and enhancing practical applications. Furthermore, the techniques developed may inspire further research in related areas of analysis and applied mathematics. This is at the core of the motivation for this article.

1.2. Contributions

This article contributes to the study of integral inequalities by introducing and exploring a new trivariate Hilbert-type integral inequality based on the following main term:

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)h(z)}{[1+(xyz)^{\alpha}]^{\lambda}}dxdydz$$

with $\alpha \in \mathbb{R}\backslash\{0\}$ and $\lambda > 0$. This integral expression is unique due to the presence of the power product term $(xyz)^{\alpha}$ in the denominator, which is also considered into a power function depending on another parameter, λ. Note that the parameter α can take both positive and negative values. This flexibility gives the derived inequality interesting properties compared to other variants, such as those presented in the previous subsection.

On this basis, four theoretical results are established, each under special parameter constraints and integrability assumptions. The first result gives a sharp and flexible upper bound on the triple integral of interest. This upper bound is expressed as a "sharp factor constant" involving the gamma function, multiplied by the following multiplication of the weighted integral norms of f, g and h:

$$\left[\int_0^{+\infty}(1+x^{\gamma})^{\beta p}\frac{f^p(x)}{x}dx\right]^{1/p}\left[\int_0^{+\infty}\left(1+y^{\delta}\right)^{\beta q}\frac{g^q(y)}{y}dy\right]^{1/q}$$
$$\times\left[\int_0^{+\infty}(1+z^{\mu})^{\beta r}\frac{h^r(z)}{z}dz\right]^{1/r},$$

where β, γ, δ and μ are parameters that can be modulated to satisfy the required integrability assumptions. The proof relies on decompositions of the triple integral, carefully introducing intermediate terms. Then the generalized Hölder integral inequality is applied, and further steps involve advanced integral calculus techniques. These elements work together to establish the sharpness of the upper bound. In the same framework, the second result concerns an inequality based on a more sophisticated triple integral, with the following expression:

$$\left[\int_0^{+\infty}(1+x^{\gamma})^{-\beta qr/(q+r)}x^{qr/[p(q+r)]}\left\{\int_0^{+\infty}\int_0^{+\infty}\frac{g(y)h(z)}{[1+(xyz)^{\alpha}]^{\lambda}}dydz\right\}^{qr/(q+r)}dx\right]^{1/q+1/r}.$$

Similar work is done for two other variants of this expression, one depending on f and g and the other on f and h. In each case, the proof is based on several integral developments and the use of the first new result under a particular configuration.

The third result complements the first by giving a new upper bound on the main triple integral. It revises the constant factor in the bound and also adjusts the different weight functions associated with the weighted integral norms of f, g and h. The proof uses changes of variables to simplify the expressions and then applies the first result under a special configuration, demonstrating the versatility of the approach.

The fourth and final result is innovative in that it gives a lower bound on the main triple integral term. This is of interest because lower bounds are often understudied in such trivariate Hilert-type integral inequality frameworks. This result ensures a more complete understanding of the previous work.

1.3. Organization of the article

The organization of the article is as follows: Section 2 presents our main result. The second and third results are given in Section 3. The proposed lower bound is discussed in Section 4. Section 5 gives a conclusion and some perspectives.

2. Main result

The main result of the article is presented in the proposition below. It specifies all the constants and assumptions of the trivariate Hilbert-type integral inequality of interest, briefly outlined in the previous section.

Proposition 2.1. *Let* $p,q,r>1$ *such that* $1/p+1/q+1/r=1$, $f,g,h:[0,+\infty)\to[0,+\infty)$, $\alpha\in\mathbb{R}\backslash\{0\}$ *and* $\lambda>0$. *Five secondary parameters are considered:* $\gamma,\delta,\mu\in\mathbb{R}\backslash\{0\}$, $\beta>0$ *and* $\nu\in\mathbb{R}$, *under the assumptions that*

$$\frac{\nu+1}{\alpha}>0,\quad \lambda\geq\frac{\nu+1}{\alpha},\quad \max\left(\frac{\nu}{\gamma},\frac{\nu}{\delta},\frac{\nu}{\mu}\right)<0,\quad \beta\geq-\min\left(\frac{\nu}{\gamma},\frac{\nu}{\delta},\frac{\nu}{\mu}\right).$$

It is also assumed that

$$0<\int_0^{+\infty}(1+x^{\gamma})^{\beta p}\frac{f^p(x)}{x}dx<+\infty,\quad 0<\int_0^{+\infty}\left(1+y^{\delta}\right)^{\beta q}\frac{g^q(y)}{y}dy<+\infty$$

and

$$0<\int_0^{+\infty}(1+z^{\mu})^{\beta r}\frac{h^r(z)}{z}dz<+\infty.$$

Then the following holds:

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{[1+(xyz)^{\alpha}]^{\lambda}}dxdydz\\
&\leq\Phi\left[\int_0^{+\infty}(1+x^{\gamma})^{\beta p}\frac{f^p(x)}{x}dx\right]^{1/p}\left[\int_0^{+\infty}\left(1+y^{\delta}\right)^{\beta q}\frac{g^q(y)}{y}dy\right]^{1/q}\\
&\times\left[\int_0^{+\infty}(1+z^{\mu})^{\beta r}\frac{h^r(z)}{z}dz\right]^{1/r},
\end{aligned}$$

where

$$\begin{aligned}
\Phi&=\frac{\Gamma((\nu+1)/\alpha)\Gamma(\lambda-(\nu+1)/\alpha)}{|\alpha|\Gamma(\lambda)}\times\frac{[\Gamma(-(\nu/\delta)p)\Gamma((\beta+\nu/\delta)p)]^{1/p}}{|\delta|^{1/p}[\Gamma(\beta p)]^{1/p}}\\
&\times\frac{[\Gamma(-(\nu/\mu)q)\Gamma((\beta+\nu/\mu)q)]^{1/q}}{|\mu|^{1/q}[\Gamma(\beta q)]^{1/q}}\times\frac{[\Gamma(-(\nu/\gamma)r)\Gamma((\beta+\nu/\gamma)r)]^{1/r}}{|\gamma|^{1/r}[\Gamma(\beta r)]^{1/r}}.
\end{aligned}\tag{2.1}$$

Proof. First, a well-known intermediate result is presented, which will be used in this proof.

Lemma 2.2. *Let* $a>-1$ *and* $b>a+1$. *Then the following holds:*

$$\int_0^{+\infty}\frac{t^a}{(1+t)^b}dt=\frac{\Gamma(a+1)\Gamma(b-a-1)}{\Gamma(b)}.$$

See [10, 3.194.3, with $\beta=1$*].*

With a judicious introduction of intermediate terms depending on the parameters involved, including the secondary parameters γ, δ, μ, β and ν, and using $1/p+1/q+1/r=1$, the main triple integral can be decomposed as follows:

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{[1+(xyz)^\alpha]^\lambda}dxdydz\\
&=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)x^{\nu/p}z^{\nu/p}}{y^{\nu(1-1/p)}[1+(xyz)^\alpha]^{\lambda/p}}\times\frac{g(y)x^{\nu/q}y^{\nu/q}}{z^{\nu(1-1/q)}[1+(xyz)^\alpha]^{\lambda/q}}\\
&\times\frac{h(z)z^{\nu/r}y^{\nu/r}}{x^{\nu(1-1/r)}[1+(xyz)^\alpha]^{\lambda/r}}dxdydz\\
&=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)x^{\nu/p}z^{\nu/p}}{y^{\nu(1-1/p)}[1+(xyz)^\alpha]^{\lambda/p}}\frac{(1+x^\gamma)^\beta}{(1+y^\delta)^\beta}\\
&\times\frac{g(y)x^{\nu/q}y^{\nu/q}}{z^{\nu(1-1/q)}[1+(xyz)^\alpha]^{\lambda/q}}\frac{(1+y^\delta)^\beta}{(1+z^\mu)^\beta}\times\frac{h(z)z^{\nu/r}y^{\nu/r}}{x^{\nu(1-1/r)}[1+(xyz)^\alpha]^{\lambda/r}}\frac{(1+z^\mu)^\beta}{(1+x^\gamma)^\beta}dxdydz.
\end{aligned}$$

From the generalized Hölder integral inequality applied with the parameters p, q and r (see [19, 5]), it follows that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)x^{\nu/p}z^{\nu/p}}{y^{\nu(1-1/p)}[1+(xyz)^\alpha]^{\lambda/p}}\frac{(1+x^\gamma)^\beta}{(1+y^\delta)^\beta}\\
&\times\frac{g(y)x^{\nu/q}y^{\nu/q}}{z^{\nu(1-1/q)}[1+(xyz)^\alpha]^{\lambda/q}}\frac{(1+y^\delta)^\beta}{(1+z^\mu)^\beta}\times\frac{h(z)z^{\nu/r}y^{\nu/r}}{x^{\nu(1-1/r)}[1+(xyz)^\alpha]^{\lambda/r}}\frac{(1+z^\mu)^\beta}{(1+x^\gamma)^\beta}dxdydz\\
&\leq A^{1/p}B^{1/q}C^{1/r},
\end{aligned}\tag{2.2}$$

where

$$A=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)x^\nu z^\nu}{y^{\nu(p-1)}[1+(xyz)^\alpha]^\lambda}\frac{(1+x^\gamma)^{\beta p}}{(1+y^\delta)^{\beta p}}dxdydz,$$

$$B=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{g^q(y)x^\nu y^\nu}{z^{\nu(q-1)}[1+(xyz)^\alpha]^\lambda}\frac{(1+y^\delta)^{\beta q}}{(1+z^\mu)^{\beta q}}dxdydz$$

and

$$C=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{h^r(z)z^\nu y^\nu}{x^{\nu(r-1)}[1+(xyz)^\alpha]^\lambda}\frac{(1+z^\mu)^{\beta r}}{(1+x^\gamma)^{\beta r}}dxdydz.$$

Using the Fubini-Tonelli integral theorem (which allows the order of integration to be exchanged, all the integrated terms being positive) and a decomposition that takes into account the variables of integration, we obtain

$$A=\int_0^{+\infty}(1+x^\gamma)^{\beta p}\frac{f^p(x)}{x}\left[\int_0^{+\infty}\frac{y^{-\nu p-1}}{(1+y^\delta)^{\beta p}}D(x,y)dy\right]dx,$$

where

$$D(x,y)=\int_0^{+\infty}\frac{(xyz)^\nu xy}{[1+(xyz)^\alpha]^\lambda}dz.$$

Applying the change of variables $t = xyz$ with respect to z, then $u = t^{\alpha}$, $\alpha \in \mathbb{R}\backslash\{0\}$, using $(\nu+1)/\alpha > 0$ and $\lambda \geq (\nu+1)/\alpha$, which allows to apply Lemma 2.2 with $a = (\nu+1)/\alpha - 1$ and $b = \lambda$, it is found that

$$D(x,y) = \int_0^{+\infty} \frac{t^{\nu}}{(1+t^{\alpha})^{\lambda}} dt = \frac{1}{|\alpha|} \int_0^{+\infty} \frac{u^{(\nu+1)/\alpha - 1}}{(1+u)^{\lambda}} du$$
$$= \frac{\Gamma((\nu+1)/\alpha)\Gamma(\lambda - (\nu+1)/\alpha)}{|\alpha|\Gamma(\lambda)}.$$

Since $D(x,y)$ is independent of x and y, A can be written as

$$A = \frac{\Gamma((\nu+1)/\alpha)\Gamma(\lambda - (\nu+1)/\alpha)}{|\alpha|\Gamma(\lambda)} \left[\int_0^{+\infty} \frac{y^{-\nu p-1}}{(1+y^{\delta})^{\beta p}} dy\right] \left[\int_0^{+\infty} (1+x^{\gamma})^{\beta p} \frac{f^p(x)}{x} dx\right].$$

For the first integral in the square brackets, applying the change of variables $v = y^{\delta}$, $\delta \in \mathbb{R}\backslash\{0\}$, using $\nu/\delta < 0$ and $\beta > -\nu/\delta$, which allows to apply Lemma 2.2 with $a = -(\nu/\delta)p - 1$ and $b = \beta p$, we obtain

$$\int_0^{+\infty} \frac{y^{-\nu p-1}}{(1+y^{\delta})^{\beta p}} dy = \frac{1}{|\delta|} \int_0^{+\infty} \frac{v^{-(\nu p+1)/\delta + 1/\delta - 1}}{(1+v)^{\beta p}} dv$$
$$= \frac{1}{|\delta|} \int_0^{+\infty} \frac{v^{-(\nu/\delta)p-1}}{(1+v)^{\beta p}} dv = \frac{\Gamma(-(\nu/\delta)p)\Gamma((\beta+\nu/\delta)p)}{|\delta|\Gamma(\beta p)}.$$

We can therefore express A as follows:

$$A = \frac{\Gamma((\nu+1)/\alpha)\Gamma(\lambda-(\nu+1)/\alpha)\Gamma(-(\nu/\delta)p)\Gamma((\beta+\nu/\delta)p)}{|\alpha||\delta|\Gamma(\lambda)\Gamma(\beta p)} \int_0^{+\infty} (1+x^{\gamma})^{\beta p} \frac{f^p(x)}{x} dx. \tag{2.3}$$

Similarly to above, analogous expressions for B and C can be found, and they are

$$B = \frac{\Gamma((\nu+1)/\alpha)\Gamma(\lambda-(\nu+1)/\alpha)\Gamma(-(\nu/\mu)q)\Gamma((\beta+\nu/\mu)q)}{|\alpha||\mu|\Gamma(\lambda)\Gamma(\beta q)} \int_0^{+\infty} \left(1+y^{\delta}\right)^{\beta q} \frac{g^q(y)}{y} dy \tag{2.4}$$

and

$$C = \frac{\Gamma((\nu+1)/\alpha)\Gamma(\lambda-(\nu+1)/\alpha)\Gamma(-(\nu/\gamma)r)\Gamma((\beta+\nu/\gamma)r)}{|\alpha||\gamma|\Gamma(\lambda)\Gamma(\beta r)} \int_0^{+\infty} (1+z^{\mu})^{\beta r} \frac{h^r(z)}{z} dz. \tag{2.5}$$

It follows from equations (2.2), (2.3), (2.4) and (2.5), and the relation $1/p + 1/q + 1/r = 1$, that

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)h(z)}{[1+(xyz)^{\alpha}]^{\lambda}} dxdydz$$
$$\leq \left[\frac{\Gamma((\nu+1)/\alpha)\Gamma(\lambda-(\nu+1)/\alpha)\Gamma(-(\nu/\delta)p)\Gamma((\beta+\nu/\delta)p)}{|\alpha||\delta|\Gamma(\lambda)\Gamma(\beta p)} \int_0^{+\infty} (1+x^{\gamma})^{\beta p} \frac{f^p(x)}{x} dx\right]^{1/p}$$
$$\times \left[\frac{\Gamma((\nu+1)/\alpha)\Gamma(\lambda-(\nu+1)/\alpha)\Gamma(-(\nu/\mu)q)\Gamma((\beta+\nu/\mu)q)}{|\alpha||\mu|\Gamma(\lambda)\Gamma(\beta q)} \int_0^{+\infty} \left(1+y^{\delta}\right)^{\beta q} \frac{g^q(y)}{y} dy\right]^{1/q}$$
$$\times \left[\frac{\Gamma((\nu+1)/\alpha)\Gamma(\lambda-(\nu+1)/\alpha)\Gamma(-(\nu/\gamma)r)\Gamma((\beta+\nu/\gamma)r)}{|\alpha||\gamma|\Gamma(\lambda)\Gamma(\beta r)} \int_0^{+\infty} (1+z^{\mu})^{\beta r} \frac{h^r(z)}{z} dz\right]^{1/r}$$
$$= \Phi \left[\int_0^{+\infty} (1+x^{\gamma})^{\beta p} \frac{f^p(x)}{x} dx\right]^{1/p} \left[\int_0^{+\infty} \left(1+y^{\delta}\right)^{\beta q} \frac{g^q(y)}{y} dy\right]^{1/q}$$

$$\times \left[\int_0^{+\infty} (1+z^{\mu})^{\beta r} \frac{h^r(z)}{z} dz\right]^{1/r},$$

where

$$\Phi = \left[\frac{\Gamma((\nu+1)/\alpha)\Gamma(\lambda-(\nu+1)/\alpha)\Gamma(-(\nu/\delta)p)\Gamma((\beta+\nu/\delta)p)}{|\alpha||\delta|\Gamma(\lambda)\Gamma(\beta p)}\right]^{1/p}$$
$$\times \left[\frac{\Gamma((\nu+1)/\alpha)\Gamma(\lambda-(\nu+1)/\alpha)\Gamma(-(\nu/\mu)q)\Gamma((\beta+\nu/\mu)q)}{|\alpha||\mu|\Gamma(\lambda)\Gamma(\beta q)}\right]^{1/q}$$
$$\times \left[\frac{\Gamma((\nu+1)/\alpha)\Gamma(\lambda-(\nu+1)/\alpha)\Gamma(-(\nu/\gamma)r)\Gamma((\beta+\nu/\gamma)r)}{|\alpha||\gamma|\Gamma(\lambda)\Gamma(\beta r)}\right]^{1/r}$$
$$= \frac{\Gamma((\nu+1)/\alpha)\Gamma(\lambda-(\nu+1)/\alpha)}{|\alpha|\Gamma(\lambda)} \times \frac{[\Gamma(-(\nu/\delta)p)\Gamma((\beta+\nu/\delta)p)]^{1/p}}{|\delta|^{1/p}[\Gamma(\beta p)]^{1/p}}$$
$$\times \frac{[\Gamma(-(\nu/\mu)q)\Gamma((\beta+\nu/\mu)q)]^{1/q}}{|\mu|^{1/q}[\Gamma(\beta q)]^{1/q}} \times \frac{[\Gamma(-(\nu/\gamma)r)\Gamma((\beta+\nu/\gamma)r)]^{1/r}}{|\gamma|^{1/r}[\Gamma(\beta r)]^{1/r}},$$

which is the formula in equation (2.1). The proof of Proposition 2.1 ends. □

Proposition 2.1 provides the theoretical foundation of a new trivariate Hilbert-type integral inequality, with all the potential applications that it generates. To highlight its interest, note that α can be positive or negative, that β, γ, δ, μ and ν are intermediate parameters that can be chosen to satisfy the integrability assumptions or to adapt to a particular mathematical scenario, and that the factor constant Φ is determined to be as sharp as possible. More classically, the norm parameters p, q and r can also be used for this purpose. Note also that the parameter α does not directly affect the weighted integral norms of f, g and h in the upper bound, thanks to the intermediate parameters.

We have thus provided a flexible framework for a sophisticated trivariate Hilbert-type integral inequality. This section is concluded by discussing a simple and complementary approach. Using $[1+(xyz)^{\alpha}]^{\lambda} \geq [(xyz)^{\alpha}]^{\lambda} = x^{\alpha\lambda}y^{\alpha\lambda}z^{\alpha\lambda}$, under the assumptions $0 < \int_0^{+\infty} x^{-\alpha\lambda} f(x)dx < +\infty$, $0 < \int_0^{+\infty} y^{-\alpha\lambda} g(y)dy < +\infty$ and $0 < \int_0^{+\infty} z^{-\alpha\lambda} h(z)dz < +\infty$, the following holds:

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)h(z)}{[1+(xyz)^{\alpha}]^{\lambda}} dxdydz$$
$$\leq \left[\int_0^{+\infty} x^{-\alpha\lambda} f(x)dx\right]\left[\int_0^{+\infty} y^{-\alpha\lambda} g(y)dy\right]\left[\int_0^{+\infty} z^{-\alpha\lambda} h(z)dz\right].$$

This inequality is valuable, but there is no flexibility in the integrability assumptions and the weighted integral norms of f, g and h. There is also no dependence on p, q and r, all of which are implicitly set to 1. Furthermore, using a direct inequality in the integral makes the factor constant suboptimal in most cases.

Proposition 2.1 is our main result, but it will also be used as an intermediate result for establishing other integral inequalities in the next sections.

3. Other new upper bound results

The proposition below adopts the framework of Proposition 2.1, and presents three new and challenging trivariate integral inequalities. Each of them depends on two functions, instead of three as in Proposition 2.1, and several parameters.

Proposition 3.1. *The same framework as in Proposition 2.1 is considered, including the factor constant Φ as given in equation (2.1).*

- *Putting $\varepsilon_{q,r} = 1/(1/q+1/r) = qr/(q+r)$, the following holds:*

$$\left[\int_0^{+\infty} (1+x^{\gamma})^{-\beta\varepsilon_{q,r}} x^{\varepsilon_{q,r}/p} \left\{\int_0^{+\infty}\int_0^{+\infty} \frac{g(y)h(z)}{[1+(xyz)^{\alpha}]^{\lambda}} dydz\right\}^{\varepsilon_{q,r}} dx\right]^{1/q+1/r}$$
$$\leq \Phi \left[\int_0^{+\infty} \left(1+y^{\delta}\right)^{\beta q} \frac{g^q(y)}{y} dy\right]^{1/q} \left[\int_0^{+\infty} (1+z^{\mu})^{\beta r} \frac{h^r(z)}{z} dz\right]^{1/r}.$$

- *Putting $\xi_{p,r} = 1/(1/p+1/r) = pr/(p+r)$, the following holds:*

$$\left[\int_0^{+\infty} \left(1+y^{\delta}\right)^{-\beta\xi_{p,r}} y^{\xi_{p,r}/q} \left\{\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)h(z)}{[1+(xyz)^{\alpha}]^{\lambda}} dxdz\right\}^{\xi_{p,r}} dy\right]^{1/p+1/r}$$
$$\leq \Phi \left[\int_0^{+\infty} (1+x^{\gamma})^{\beta p} \frac{f^p(x)}{x} dx\right]^{1/p} \left[\int_0^{+\infty} (1+z^{\mu})^{\beta r} \frac{h^r(z)}{z} dz\right]^{1/r}.$$

- *Putting $\zeta_{p,q} = 1/(1/p+1/q) = pq/(p+q)$, the following holds:*

$$\left[\int_0^{+\infty} (1+z^{\mu})^{-\beta\zeta_{p,q}} z^{\zeta_{p,q}/r} \left\{\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{[1+(xyz)^{\alpha}]^{\lambda}} dxdy\right\}^{\zeta_{p,q}} dz\right]^{1/p+1/q}$$
$$\leq \Phi \left[\int_0^{+\infty} (1+x^{\gamma})^{\beta p} \frac{f^p(x)}{x} dx\right]^{1/p} \left[\int_0^{+\infty} \left(1+y^{\delta}\right)^{\beta q} \frac{g^q(y)}{y} dy\right]^{1/q}.$$

Proof. For the sake of redundancy, we will focus on the first point; the other two can be proved in a similar way. Let us set

$$F = \int_0^{+\infty} (1+x^{\gamma})^{-\beta\varepsilon_{q,r}} x^{\varepsilon_{q,r}/p} \left\{\int_0^{+\infty}\int_0^{+\infty} \frac{g(y)h(z)}{[1+(xyz)^{\alpha}]^{\lambda}} dydz\right\}^{\varepsilon_{q,r}} dx.$$

This term will appear later in the proof in the form of an upper bound, as an intermediate step to the establishment of the desired result.

Using the Fubini-Tonelli integral theorem, a suitable decomposition of F gives

$$\begin{aligned} F &= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} (1+x^{\gamma})^{-\beta\varepsilon_{q,r}} x^{\varepsilon_{q,r}/p} \frac{g(y)h(z)}{[1+(xyz)^{\alpha}]^{\lambda}} \\ &\quad \times \left\{\int_0^{+\infty}\int_0^{+\infty} \frac{g(y)h(z)}{[1+(xyz)^{\alpha}]^{\lambda}} dydz\right\}^{\varepsilon_{q,r}-1} dxdydz \\ &= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{f_{\star}(x)g(y)h(z)}{[1+(xyz)^{\alpha}]^{\lambda}} dxdydz, \end{aligned} \tag{3.1}$$

where

$$f_{\star}(x) = (1+x^{\gamma})^{-\beta\varepsilon_{q,r}} x^{\varepsilon_{q,r}/p} \left\{\int_0^{+\infty}\int_0^{+\infty} \frac{g(y)h(z)}{[1+(xyz)^{\alpha}]^{\lambda}} dydz\right\}^{\varepsilon_{q,r}-1}.$$

Proposition 2.1 applied with $f = f_\star$ gives

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{f_\star(x)g(y)h(z)}{[1+(xyz)^\alpha]^\lambda}dxdydz \\
&\leq \Phi\left[\int_0^{+\infty}(1+x^\gamma)^{\beta p}\frac{f_\star^p(x)}{x}dx\right]^{1/p}\left[\int_0^{+\infty}\left(1+y^\delta\right)^{\beta q}\frac{g^q(y)}{y}dy\right]^{1/q} \\
&\times\left[\int_0^{+\infty}(1+z^\mu)^{\beta r}\frac{h^r(z)}{z}dz\right]^{1/r}. \qquad (3.2)
\end{aligned}$$

If we focus on the first term in the square brackets, we have

$$\begin{aligned}
&\int_0^{+\infty}(1+x^\gamma)^{\beta p}\frac{f_\star^p(x)}{x}dx \\
&= \int_0^{+\infty}(1+x^\gamma)^{\beta p}\frac{1}{x}(1+x^\gamma)^{-\beta p\varepsilon_{q,r}}x^{\varepsilon_{q,r}}\left\{\int_0^{+\infty}\int_0^{+\infty}\frac{g(y)h(z)}{[1+(xyz)^\alpha]^\lambda}dydz\right\}^{p(\varepsilon_{q,r}-1)}dx \\
&= \int_0^{+\infty}(1+x^\gamma)^{\beta p(1-\varepsilon_{q,r})}x^{\varepsilon_{q,r}-1}\left\{\int_0^{+\infty}\int_0^{+\infty}\frac{g(y)h(z)}{[1+(xyz)^\alpha]^\lambda}dydz\right\}^{p(\varepsilon_{q,r}-1)}dx. \qquad (3.3)
\end{aligned}$$

Now, using the relation $1/p+1/q+1/r = 1$, so that $\varepsilon_{q,r} = 1/(1/q+1/r) = 1/(1-1/p)$, the following holds:

$$p(\varepsilon_{q,r}-1) = p\left(\frac{1}{1-1/p}-1\right) = p\frac{1/p}{1-1/p} = \frac{1}{1-1/p} = \frac{1}{1/q+1/r} = \varepsilon_{q,r}.$$

We therefore have

$$\begin{aligned}
&\int_0^{+\infty}(1+x^\gamma)^{\beta p(1-\varepsilon_{q,r})}x^{\varepsilon_{q,r}-1}\left\{\int_0^{+\infty}\int_0^{+\infty}\frac{g(y)h(z)}{[1+(xyz)^\alpha]^\lambda}dydz\right\}^{p(\varepsilon_{q,r}-1)}dx \\
&= \int_0^{+\infty}(1+x^\gamma)^{-\beta\varepsilon_{q,r}}x^{\varepsilon_{q,r}/p}\left\{\int_0^{+\infty}\int_0^{+\infty}\frac{g(y)h(z)}{[1+(xyz)^\alpha]^\lambda}dydz\right\}^{\varepsilon_{q,r}}dx \\
&= F. \qquad (3.4)
\end{aligned}$$

The combination of the equations (3.1), (3.2), (3.3) and (3.4) results in

$$F \leq \Phi\times F^{1/p}\left[\int_0^{+\infty}\left(1+y^\delta\right)^{\beta q}\frac{g^q(y)}{y}dy\right]^{1/q}\left[\int_0^{+\infty}(1+z^\mu)^{\beta r}\frac{h^r(z)}{z}dz\right]^{1/r},$$

so that

$$F^{1-1/p} \leq \Phi\left[\int_0^{+\infty}\left(1+y^\delta\right)^{\beta q}\frac{g^q(y)}{y}dy\right]^{1/q}\left[\int_0^{+\infty}(1+z^\mu)^{\beta r}\frac{h^r(z)}{z}dz\right]^{1/r}.$$

Using the expression of F and the fact that $1-1/p = 1/q+1/r$, the above inequality can also be written as follows:

$$\begin{aligned}
&\left[\int_0^{+\infty}(1+x^\gamma)^{-\beta\varepsilon_{q,r}}x^{\varepsilon_{q,r}/p}\left\{\int_0^{+\infty}\int_0^{+\infty}\frac{g(y)h(z)}{[1+(xyz)^\alpha]^\lambda}dydz\right\}^{\varepsilon_{q,r}}dx\right]^{1/q+1/r} \\
&\leq \Phi\left[\int_0^{+\infty}\left(1+y^\delta\right)^{\beta q}\frac{g^q(y)}{y}dy\right]^{1/q}\left[\int_0^{+\infty}(1+z^\mu)^{\beta r}\frac{h^r(z)}{z}dz\right]^{1/r}.
\end{aligned}$$

The desired inequality is established. This ends the proof of Proposition 3.1. □

Proposition 3.1 can be used as an integral tool in functional analysis, and especially in operator theory, where weighted integral norms of various complex functions appear.

The proposition below offers an alternative approach to that of Proposition 2.1, with different assumptions and different weighted integral norms involved in the upper bound. The main difference in the proof is a direct change of variables in the main triple integral.

Proposition 3.2. *Let $p,q,r>1$ such that $1/p+1/q+1/r=1$, $f,g,h:[0,+\infty)\to[0,+\infty)$, $\alpha\in\mathbb{R}\backslash\{0\}$ and $\lambda>0$. Five secondary parameters are considered: $\gamma,\delta,\mu\in\mathbb{R}\backslash\{0\}$, $\beta>0$ and $\nu\in\mathbb{R}$, under the assumptions that*

$$\nu>-1,\quad \lambda\geq\nu+1,\quad \max\left(\frac{\nu}{\gamma},\frac{\nu}{\delta},\frac{\nu}{\mu}\right)<0,\quad \beta\geq-\min\left(\frac{\nu}{\gamma},\frac{\nu}{\delta},\frac{\nu}{\mu}\right).$$

It is also assumed that

$$0<\int_0^{+\infty}(1+x^{\alpha\gamma})^{\beta p}x^{p(1-\alpha)-1}f^p(x)dx<+\infty,$$

$$0<\int_0^{+\infty}\left(1+y^{\alpha\delta}\right)^{\beta q}y^{q(1-\alpha)-1}g^q(y)dy<+\infty$$

and

$$0<\int_0^{+\infty}(1+z^{\alpha\mu})^{\beta r}z^{r(1-\alpha)-1}h^r(z)dz<+\infty.$$

Then the following holds:

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{[1+(xyz)^{\alpha}]^{\lambda}}dxdydz\\&\leq\frac{1}{|\alpha|^2}\Psi\left[\int_0^{+\infty}(1+x^{\alpha\gamma})^{\beta p}x^{p(1-\alpha)-1}f^p(x)dx\right]^{1/p}\\&\times\left[\int_0^{+\infty}\left(1+y^{\alpha\delta}\right)^{\beta q}y^{q(1-\alpha)-1}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z^{\alpha\mu})^{\beta r}z^{r(1-\alpha)-1}h^r(z)dz\right]^{1/r},\end{aligned}$$

where Ψ is given in equation (2.1) with $\alpha=1$, i.e.,

$$\begin{aligned}\Psi=&\frac{\Gamma(\nu+1)\Gamma(\lambda-(\nu+1))}{\Gamma(\lambda)}\times\frac{[\Gamma(-(\nu/\delta)p)\Gamma((\beta+\nu/\delta)p)]^{1/p}}{|\delta|^{1/p}[\Gamma(\beta p)]^{1/p}}\\&\times\frac{[\Gamma(-(\nu/\mu)q)\Gamma((\beta+\nu/\mu)q)]^{1/q}}{|\mu|^{1/q}[\Gamma(\beta q)]^{1/q}}\times\frac{[\Gamma(-(\nu/\gamma)r)\Gamma((\beta+\nu/\gamma)r)]^{1/r}}{|\gamma|^{1/r}[\Gamma(\beta r)]^{1/r}}.\end{aligned}$$

Proof. Based on the main triple integral, the changes of variables $u=x^{\alpha}$, $v=y^{\alpha}$ and $w=z^{\alpha}$ give

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{[1+(xyz)^{\alpha}]^{\lambda}}dxdydz\\&=\frac{1}{|\alpha|^3}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(u^{1/\alpha})g(v^{1/\alpha})h(w^{1/\alpha})}{(1+uvw)^{\lambda}}u^{1/\alpha-1}v^{1/\alpha-1}w^{1/\alpha-1}dudvdw\\&=\frac{1}{|\alpha|^3}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f_{\dagger}(u)g_{\dagger}(v)h_{\dagger}(w)}{(1+uvw)^{\lambda}}dudvdw,\end{aligned}\tag{3.5}$$

where

$$f_\dagger(u)=f(u^{1/\alpha})u^{1/\alpha-1},\quad g_\dagger(v)=g(v^{1/\alpha})v^{1/\alpha-1},\quad h_\dagger(w)=h(w^{1/\alpha})w^{1/\alpha-1}.$$

Applying Proposition 2.1 with $\alpha=1$, $f=f_\dagger$, $g=g_\dagger$ and $h=h_\dagger$, we obtain

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f_\dagger(u)g_\dagger(v)h_\dagger(w)}{(1+uvw)^\lambda}dudvdw\\&\leq\Psi\left[\int_0^{+\infty}(1+u^\gamma)^{\beta p}\frac{f_\dagger^p(u)}{u}du\right]^{1/p}\left[\int_0^{+\infty}\left(1+v^\delta\right)^{\beta q}\frac{g_\dagger^q(v)}{v}dv\right]^{1/q}\\&\times\left[\int_0^{+\infty}(1+w^\mu)^{\beta r}\frac{h_\dagger^r(w)}{w}dw\right]^{1/r}.\end{aligned}\tag{3.6}$$

For the first term in the square brackets, using the definition of $f_\dagger$ and the change of variables $u=x^\alpha$, the result is

$$\begin{aligned}&\int_0^{+\infty}(1+u^\gamma)^{\beta p}\frac{f_\dagger^p(u)}{u}du=\int_0^{+\infty}(1+u^\gamma)^{\beta p}\frac{f^p(u^{1/\alpha})u^{p(1/\alpha-1)}}{u}du\\&=|\alpha|\int_0^{+\infty}(1+x^{\alpha\gamma})^{\beta p}\frac{f^p(x)x^{p(1-\alpha)}}{x^\alpha}x^{\alpha-1}dx\\&=|\alpha|\int_0^{+\infty}(1+x^{\alpha\gamma})^{\beta p}x^{p(1-\alpha)-1}f^p(x)dx.\end{aligned}\tag{3.7}$$

Similarly to above, it is found that

$$\int_0^{+\infty}\left(1+v^\delta\right)^{\beta q}\frac{g_\dagger^q(v)}{v}dv=|\alpha|\int_0^{+\infty}\left(1+y^{\alpha\delta}\right)^{\beta q}y^{q(1-\alpha)-1}g^q(y)dy\tag{3.8}$$

and

$$\int_0^{+\infty}(1+w^\mu)^{\beta r}\frac{h_\dagger^r(w)}{w}dw=|\alpha|\int_0^{+\infty}(1+z^{\alpha\mu})^{\beta r}z^{r(1-\alpha)-1}h^r(z)dz.\tag{3.9}$$

Combining equations (3.5), (3.6), (3.7), (3.8) and (3.9) and using $1/p+1/q+1/r=1$, we get

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{[1+(xyz)^\alpha]^\lambda}dxdydz\\&\leq\frac{1}{|\alpha|^3}\Psi\left[|\alpha|\int_0^{+\infty}(1+x^{\alpha\gamma})^{\beta p}x^{p(1-\alpha)-1}f^p(x)dx\right]^{1/p}\\&\times\left[|\alpha|\int_0^{+\infty}\left(1+y^{\alpha\delta}\right)^{\beta q}y^{q(1-\alpha)-1}g^q(y)dy\right]^{1/q}\left[|\alpha|\int_0^{+\infty}(1+z^{\alpha\mu})^{\beta r}z^{r(1-\alpha)-1}h^r(z)dz\right]^{1/r}\\&=\frac{1}{|\alpha|^2}\Psi\left[\int_0^{+\infty}(1+x^{\alpha\gamma})^{\beta p}x^{p(1-\alpha)-1}f^p(x)dx\right]^{1/p}\\&\times\left[\int_0^{+\infty}\left(1+y^{\alpha\delta}\right)^{\beta q}y^{q(1-\alpha)-1}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}(1+z^{\alpha\mu})^{\beta r}z^{r(1-\alpha)-1}h^r(z)dz\right]^{1/r}.\end{aligned}$$

The desired inequality is obtained. The proof of Proposition 3.2 ends. □

The results in Propositions 2.1 and 3.2 are similar but difficult to compare because they use different weighted integral norms and are made under different assumptions. Note only that the parameter α in these norms has a more direct effect in Proposition 3.2 than in Proposition 2.1. Furthermore, Proposition 3.2 can be seen as a consequence, since it was established via Proposition 2.1. So these propositions are complementary.

4. A new lower bound result

To complete the work in the previous sections, the proposition below gives a lower bound on the main triple integral term. This is of interest because, overall, such a lower bound remains an understudied aspect in the field of Hilbert-type integral inequalities.

Proposition 4.1. *Let $f,g,h:[0,+\infty)\to[0,+\infty)$ and $\alpha,\lambda\in\mathbb{R}$. It is assumed that*

$$0<\int_0^{+\infty} f(x)dx<+\infty, \quad 0<\int_0^{+\infty} g(y)dy<+\infty, \quad 0<\int_0^{+\infty} h(z)dz<+\infty,$$

$$0<\int_0^{+\infty}(1+x^\alpha)^\lambda f(x)dx<+\infty, \quad 0<\int_0^{+\infty}(1+y^\alpha)^\lambda g(y)dy<+\infty$$

and

$$0<\int_0^{+\infty}(1+z^\alpha)^\lambda h(z)dz<+\infty.$$

Then the following holds:

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{[1+(xyz)^\alpha]^\lambda}dxdydz\\ &\geq\frac{\left[\int_0^{+\infty}f(x)dx\right]^2\left[\int_0^{+\infty}g(y)dy\right]^2\left[\int_0^{+\infty}h(z)dz\right]^2}{\left[\int_0^{+\infty}(1+x^\alpha)^\lambda f(x)dx\right]\left[\int_0^{+\infty}(1+y^\alpha)^\lambda g(y)dy\right]\left[\int_0^{+\infty}(1+z^\alpha)^\lambda h(z)dz\right]}.\end{aligned}$$

Proof. Using a well-configured Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\left[\int_0^{+\infty}f(x)dx\right]\left[\int_0^{+\infty}g(y)dy\right]\left[\int_0^{+\infty}h(z)dz\right]=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}f(x)g(y)h(z)dxdydz\\ &=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{[f(x)g(y)h(z)]^{1/2}}{[1+(xyz)^\alpha]^{\lambda/2}}[1+(xyz)^\alpha]^{\lambda/2}[f(x)g(y)h(z)]^{1/2}dxdydz\\ &\leq\left\{\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}[1+(xyz)^\alpha]^\lambda f(x)g(y)h(z)dxdydz\right\}^{1/2}\\ &\times\left\{\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{[1+(xyz)^\alpha]^\lambda}dxdydz\right\}^{1/2},\end{aligned}$$

so that

$$\begin{aligned}&\left[\int_0^{+\infty}f(x)dx\right]^2\left[\int_0^{+\infty}g(y)dy\right]^2\left[\int_0^{+\infty}h(z)dz\right]^2\\ &\leq\left\{\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}[1+(xyz)^\alpha]^\lambda f(x)g(y)h(z)dxdydz\right\}\\ &\times\left\{\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{[1+(xyz)^\alpha]^\lambda}dxdydz\right\}.\end{aligned}\tag{4.1}$$

The standard inequality $1+abc\leq(1+a)(1+b)(1+c)$ with $a,b,c\geq 0$ implies that

$$1+(xyz)^\alpha=1+x^\alpha y^\alpha z^\alpha\leq(1+x^\alpha)(1+y^\alpha)(1+z^\alpha).$$

This, in combination with the obtained separability with respect to x, y and z, gives

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}[1+(xyz)^{\alpha}]^{\lambda} f(x)g(y)h(z)dxdydz \\
&\leq \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}(1+x^{\alpha})^{\lambda}(1+y^{\alpha})^{\lambda}(1+z^{\alpha})^{\lambda} f(x)g(y)h(z)dxdydz \\
&= \left[\int_0^{+\infty}(1+x^{\alpha})^{\lambda} f(x)dx\right]\left[\int_0^{+\infty}(1+y^{\alpha})^{\lambda} g(y)dy\right]\left[\int_0^{+\infty}(1+z^{\alpha})^{\lambda} h(z)dz\right]. \qquad (4.2)
\end{aligned}$$

Combining equations (4.1) and (4.2), we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{[1+(xyz)^{\alpha}]^{\lambda}}dxdydz \\
&\geq \frac{\left[\int_0^{+\infty} f(x)dx\right]^2\left[\int_0^{+\infty} g(y)dy\right]^2\left[\int_0^{+\infty} h(z)dz\right]^2}{\left[\int_0^{+\infty}(1+x^{\alpha})^{\lambda} f(x)dx\right]\left[\int_0^{+\infty}(1+y^{\alpha})^{\lambda} g(y)dy\right]\left[\int_0^{+\infty}(1+z^{\alpha})^{\lambda} h(z)dz\right]}.
\end{aligned}$$

The desired inequality is established. This ends the proof of Proposition 4.1. □

Note that, in this proposition, there is no restriction on the parameters α and λ; we can use $\alpha, \lambda \in \mathbb{R}$, provided that the minimal integration assumptions are satisfied.

Therefore, under certain integrability assumptions, the upper bounds obtained in Propositions 2.1, 3.1 and 3.2 cannot be smaller than the lower bound given in Proposition 4.1. This allows us to fully understand the most important mathematical facets of the triple integral under consideration.

5. Conclusion and perspectives

This article mainly studies in detail a new flexible trivariate Hilbert-type integral inequality. Several upper bounds are obtained under well-identified assumptions on the functions and parameters involved. Sharp factor constants depending on the gamma function are shown. This is complemented by the determination of a lower bound for the main triple integral. Perspectives of this work include the consideration of the finite interval integration version of our main inequality, i.e., depending on the following main term:

$$\int_0^{a}\int_0^{a}\int_0^{a}\frac{f(x)g(y)h(z)}{[1+(xyz)^{\alpha}]^{\lambda}}dxdydz$$

with $a > 0$. The following variant can also be considered:

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{|1-(xyz)^{\alpha}|^{\lambda}}dxdydz,$$

which is inspired by the study in [17], or to a higher dimensional study, depending on the following main term:

$$\int_0^{+\infty}\dots\int_0^{+\infty}\frac{\prod_{i=1}^{n} f_i(x_i)}{[1+\prod_{i=1}^{n} x_i^{\alpha_i}]^{\lambda}}dx_1\dots dx_n$$

with $f_1,\dots,f_n : [0,+\infty) \to [0,+\infty)$ and $\alpha_1,\dots,\alpha_n \in \mathbb{R}\backslash\{0\}$. Clearly, the multiple dimensions add considerable complexity to the problem. These studies are left for the future.

Acknowledgement

The author would like to thank the four reviewers for their constructive comments on the article.

References

[1] V. Adiyasuren, T. Batbold, and M. Krnić, *Multiple Hilbert-type inequalities involving some differential operators*, Banach Journal of Mathematical Analysis, 10(2), 2016, 320–337. DOI: 10.1215/17358787-3495561

[2] V. Adiyasuren, T. Batbold, and M. Krnić, *Hilbert-type inequalities involving differential operators, the best constants and applications*, Mathematical Inequalities & Applications, 18(1), 2015, 111–124. DOI: 10.7153/mia-18-07

[3] L. E. Azar, *The connection between Hilbert and Hardy inequalities*, Journal of Inequalities and Applications, 2013, Art. 452, 1–10. DOI: 10.1186/1029-242X-2013-452

[4] T. Batbold and Y. Sawano, *Sharp bounds for m-linear Hilbert-type operators on the weighted Morrey spaces*, Mathematical Inequalities & Applications, 20(1), 2017, 263–283. DOI: 10.7153/mia-20-20

[5] B. Benaissa and M. Z. Sarikaya, *On the refinements of some important inequalities with a finite set of positive numbers*, Mathematical Methods in the Applied Sciences, 47, 2024, 9589–9599. DOI: 10.1002/mma.10084

[6] A. Bényi and C. T. Oh, *Best constant for certain multilinear integral operator*, Journal of Inequalities and Applications, 2006, Art. 28582, 1–12. DOI: 10.1155/JIA/2006/28582

[7] Q. Chen and B. C. Yang, *A survey on the study of Hilbert-type inequalities*, Journal of Inequalities and Applications, 2015, Art. 302, 1–29. DOI: 10.1186/s13660-015-0829-7

[8] C. Chesneau, *Some four-parameter trigonometric generalizations of the Hilbert integral inequality*, Asia Mathematika, 8(2), 2024, 45–59. DOI: 10.5281/zenodo.13949386

[9] H. Du and Y. Miao, *Several new Hardy-Hilbert's inequalities*, Filomat, 25(3), 2011, 153–162. DOI: 10.2298/FIL1103153D

[10] I. S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, 7th Edition, Academic Press, 2007.

[11] G. H. Hardy, J.E. Littlewood, and G. Polya, Inequalities, Cambridge University Press, Cambridge, 1934.

[12] Y. Hong, *On multiple Hardy-Hilbert integral inequalities with some parameters*, Journal of Inequalities and Applications 2006, Art. 94960 1–11. DOI: 10.1155/JIA/2006/94960

[13] Z. Huang and B. C. Yang, *A multidimensional Hilbert-type integral inequality*, Journal of Inequalities and Applications, 2015, Art. 151, 1–13. DOI: 10.1186/s13660-015-0673-9

[14] S. W. Jian and F.Z. Yang, *All-sided generalization about Hardy-Hilbert integral inequalities*, Acta Mathematica Sinica (China), 44(4), 2001, 619–626. DOI:10.12386/A2001sxxb0081

[15] Y. Li, Y. Qian, and B. He, *On further analogs of Hilbert's inequality*, International Journal of Mathematics and Mathematical Sciences, 2007, Art. 76329, 1–6. DOI: 10.1155/2007/76329

[16] W. T. Sulaiman, *On Hardy-Hilbert's integral inequality*, Journal of Inequalities in Pure and Applied Mathematics, 5(2), 2004, 1–9.

[17] W. T. Sulaiman, *New types of Hardy-Hilbert's integral inequality*, General Mathematics Notes, 2(2), 2011, 111–118.

[18] B. Sun, *A multiple Hilbert-type integral inequality with the best constant factor*, Journal of Inequalities and Applications, 2007, Art. 71049, 1–14. DOI: 10.1155/2007/71049

[19] J. F. Tian, *Properties of generalized Hölder's inequalities*, Journal of Mathematical Inequalities, 9(2), 2015, 473–480. DOI: 10.7153/jmi-09-40

[20] D. C. Ullrich, *A simple elementary proof of Hilbert's inequality*, The American Mathematical Monthly, 120(2), 2013, 161–164. DOI: 10.4169/amer.math.monthly.120.02.161

[21] J. S. Xu, *Hardy-Hilbert's inequalities with two parameters*, Advances in Mathematics, 36(2), 2007, 63–76.

[22] B. C. Yang, *On Hilbert's integral inequality*, Journal of Mathematical Analysis and Applications, 220(2), 1998, 778–785. DOI: 10.1006/jmaa.1997.5877

[23] B. C. Yang, *A multiple Hardy-Hilbert integral inequality*, Chinese Annals of Mathematics. Series A, 24(6), 2003, 743–750.

[24] B. C. Yang, *On the norm of an integral operator and applications*, Journal of Mathematical Analysis and Applications, 321(1), 2006, 182–192. DOI: 10.1016/j.jmaa.2005.07.071

[25] B. C. Yang, *On the norm of a Hilbert's type linear operator and applications*, Journal of Mathematical Analysis and Applications, 325(1), 2007, 529–541. DOI: 10.1016/j.jmaa.2006.02.006

[26] B. C. Yang, *The Norm of Operator and Hilbert-Type Inequalities*, Science Press, Beijing, 2009.

[27] B. C. Yang, *Hilbert-Type Integral Inequalities*, Bentham Science Publishers, The United Arab Emirates, 2009.

[28] B. C. Yang, and M. Krnić, *On the norm of a multi-dimensional Hilbert-type operator*, Sarajevo Journal of Mathematics, 7(2), 2011, 223–243. 10.5644/SJM.07.2.08

[29] W. Y. Zhong, and B. C. Yang, *On a multiple Hilbert-type integral inequality with the symmetric kernel*, Journal of Inequalities and Applications, 2007, Art. 27962, 1–17. DOI: 10.1155/2007/27962